Small time asymptotics for an example of strictly hypoelliptic heat kernel

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Abstract

A small time asymptotics of the density is established for a simplified (non-Gaussian, strictly hypoelliptic) second chaos process tangent to the Dudley relativistic diffusion.

1 Introduction

The problem of estimating the heat kernel, or the density of a diffusion, particularly as time goes to zero, has been extensively studied for a long time. Let us mention only the articles [V], [A], [BA1], [BA2], [L], [ERS], and the existence of other works on that subject by Azencott, Molchanov and Bismut, quoted in [BA1].

To summary roughly, a very classical question addresses the asymptotic behavior (as $s \searrow 0$) of the density $p_s(x,y)$ of the diffusion (x_s) solving a Stratonovich stochastic differential equation

$$x_s = x + \sum_{j=1}^{k} \int_0^s V_j(x_\tau) \circ dW_\tau^j + \int_0^s V_0(x_\tau)\, d\tau\,,$$

where the smooth vector fields V_j are supposed to satisfy a Hörmander condition ; the underlying space being $\mathbb{R}^d$ or some d-dimensional smooth manifold $\mathcal{M}$.

The elliptic case being very well understood for a long time ([V], [A]), the studies focussed then on the sub-elliptic case, that is to say, when the strong Hörmander condition (that the Lie algebra generated by the fields $V_1, \ldots, V_k$ has maximal rank everywhere) is fulfilled. In that case these fields generate a sub-Riemannian distance $d(x,y)$, defined as in control theory, by considering only C^1 paths whose tangent vectors are spanned by them. Then the wanted asymptotic expansion tends to have the following Gaussian-like form:

$$p_s(x,y) = s^{-d/2} \exp\big(-d(x,y)^2/(2s)\big)\left(\sum_{\ell=0}^{n} \gamma_\ell(x,y)\, s^\ell + \mathcal{O}(s^{n+1})\right) \tag{1}$$

for any $n \in \mathbb{N}^*$, with smooth γ_ℓ's and $\gamma_0 > 0$, provided (x,y) does not belong to the cut-locus (and uniformly within any compact set which does not intersect the cut-locus). See in particular ([BA1], théorème 3.1). Note that the condition of remaining outside the cut-locus is here necessary, as showed in particular by [BA2].

The methods used to get this or a similar result have been of different nature. In [BA1], G. Ben Arous proceeds by expanding the flow associated to the diffusion (in this direction, see also [Ca]) and using a Laplace method applied to the Fourier transform of x_s, then inverted by means of Malliavin's calculus.

The strictly hypoelliptic case, i.e., when only the weak Hörmander condition (requiring the use of the drift vector field V_0 to recover the full tangent space) is fulfilled, remains much more problematic, and then rarely addressed. There is a priori no longer any reason that in such case the asymptotic behavior of $p_s(x,y)$ remains of the Gaussian-like type (1), all the less as a natural candidate for replacing the sub-Riemannian distance $d(x,y)$ is missing. Indeed this already fails for the mere (however Gaussian) Langevin process $\left(\beta_s, \int_0^s \beta_\tau d\tau\right)$: the missing distance is replaced by a time-dependent distance which presents some degeneracy in one direction, namely $\frac{6}{s^3}\left|(x-y) - \frac{s}{2}(\dot{x}-\dot{y})\right|^2 + \frac{1}{2s}|\dot{x}-\dot{y}|^2$, see (11), (13) below. See also [DM] for a more involved (non-curved, strictly hypoelliptic, perturbed) case where Langevin-like estimates hold (without precise asymptotics), roughly having the following Li-Yau-like form:

$$C^{-1}\, s^{-N}\, e^{-C\, d_s(x_s,y)^2} \,\leq\, p_s(x,y) \,\leq\, C\, s^{-N}\, e^{-C^{-1}\, d_s(x_s,y)^2}\,, \quad \text{for } \; 0 < s < s_0\,. \tag{2}$$

In this article, an interesting case of rather natural hypoelliptic diffusion is considered first: that of a relativistic diffusion, first constructed in Minkowski's space (see [Du]), which makes sense on a generic smooth Lorentzian manifold, see [F-LJ1], [F-LJ2], [F-LJ3]. In the simplest case of Minkowski's space, it consists in the pair $(\xi_s, \dot{\xi}_s) \in \mathbb{R}^{1,d} \times \mathbb{H}^d$ (parametrized by its proper time s, and analogous to a Langevin process), where the velocity $(\dot{\xi}_s)$ is a hyperbolic Brownian motion. In the general case this Dudley diffusion can be rolled without slipping from a reference tangent space to the Lorentzian manifold, see [F-LJ1]. Note that even in the Minkowski space, there is a curvature constraint to be taken into account, namely that of the mass shell $\mathbb{H}^d$, at the heart of this framework. Moreover the relativistic diffusion is never sub-elliptic, but only hypoelliptic, and a priori a Gaussian-like asymptotic expansion as (1) does not even make sense, since there is no longer any natural candidate to replace the sub-Riemannian distance $d(x,y)$. See however [BF], where some non-trivial information is extracted about the relativistic diffusion, by considering the sub-Riemannian distance generated by all fields $V_0, V_1, \ldots, V_k$ (i.e., not only $V_1, \ldots, V_k$). Talking of this, an important feature of the strictly hypoelliptic case, which is fulfilled in the relativistic framework, is when the graded geometry generated by the successive brackets of a given weight is (at least locally) constant (see [NSW], and also [T]), yielding homogeneity in the afore-mentioned time-dependent distance.

To proceed, we shall compute a Fourier-Laplace transform, which seems to be the only way of getting any quantitative access to the density kernel of the relativistic diffusion

([BA1] relies already on the Fourier transform, but then the method followed by G. Ben Arous is based on a stochastic variation about a minimal geodesic, which does not exist here, and on the local strict convexity of the energy functional (due to the sub-ellipticity), which does not hold here). Because of the singularity in the most natural polar coordinates, we first choose alternative, less intuitive but smooth coordinates, and using them, partially expand the relativistic diffusion to project it on the second Wiener chaos, thereby exhibiting a simplified "tangent process". For this simplified process the Fourier-Laplace transform is exactly computable. Then analyzing its inverse Fourier transform very carefully and using a saddle-point method allows to derive an asymptotic equivalent for the density q_s of this tangent process, as time s goes to zero (see Theorem 6.3 below). As in the Langevin or in the more sophisticated case of [DM], the exponential term is given by a time-dependent distance, namely the same as the afore-mentioned Langevin one, the strictly second chaos coordinate appearing only in the off-exponent term. The initial analogous question about the relativistic diffusion remains open, as the degree of contact between both considered processes (the effective computation of the Fourier-Laplace transform being bounded to the second Wiener chaos) seems so far too weak, to allow to deduce a former asymptotic behaviour from the second one. The non-appearance of the non-Gaussian coordinates in the found asymptotic exponent lets however think that this could remain so for all higher order chaos terms of the Taylor expansion. Thence a tempting guess, resulting from both the sub-elliptic case (1) as solved by [BA1], the modified Li-Yau-like estimates (2) obtained in [DM] and the present work, whose main result is Theorem 6.3 below, is that an expansion having the following form could (maybe generally, under consistency of the Lie graded geometry) hold:

$$p_s(x,y) = s^{-N} \exp\Big(- d_s(x,y)^2\Big)\bigg(\sum_{\ell=0}^{n} \gamma_\ell(x,y)\, s^\ell + \mathcal{O}(s^{n+1})\bigg).$$

In this first attempt we restrict to the simplest case of the five-dimensional Minkowski space $\mathbb{R}^{1,2} \times \mathbb{H}^2$ and then to its five-dimensional second chaos tangent process. The case of the generic Minkowski space $\mathbb{R}^{1,d} \times \mathbb{H}^d$ is actually very analogous, but would mainly bring notational difficulties without modifying the method. We hope that this particular toy example will allow to understand better what can happen and could be undertaken, concerning small time asymptotics of the relativistic diffusion itself, and then maybe in some more generic strictly hypoelliptic framework.

The content is organized as follows.

In Section 2 are mainly described the setting and the smooth parametrization used then. In Section 3 the simplified "tangent process" (Y_s) to the relativistic diffusion (X_s) is exhibited. In Section 4 the Fourier-Laplace transform of the tangent process (Y_s) is computed. In Section 5 a closed integral expression for the density q_s of (Y_s) is explicited. Section 6 yields a precise (off the diagonal) equivalent for the density q_s as $s \to 0$, which is the content of the main result (Theorem 6.3). Section 7 contains three rather technical proofs, which have been postponed till there to lighten the reading.

2 A smooth parametrization of $\mathbb{H}^2 \times \mathbb{R}^{1,2} \equiv T^1_+\mathbb{R}^{1,2}$

The Dudley relativistic diffusion $X_s = (\dot\xi_s, \xi_s)$ (see [Du]) lives in the future-directed unit tangent bundle $T^1_+\mathbb{R}^{1,2} \equiv \mathbb{H}^2 \times \mathbb{R}^{1,2}$ to the Minkowski space $\big(\mathbb{R}^{1,2}, \langle\cdot,\cdot\rangle\big)$ $\big($or alternatively, in its frame bundle, isomorphic to the Poincaré isometry group $\mathcal{P}^3 = \mathrm{PSO}(1,2) \propto \mathbb{R}^{1,2}\big)$. We classically identify the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{R}^{1,2}$ with the upper sheet of the hyperboloid having equation $|\dot\xi^0|^2 - |\dot\xi^1|^2 - |\dot\xi^2|^2 \equiv \langle\dot\xi,\dot\xi\rangle = 1$ within $\mathbb{R}^{1,2}$ $\big($endowed with its canonical basis $(e_0,e_1,e_2)\big)$. The velocity sub-diffusion $(\dot\xi_s)$ is a hyperbolic Brownian motion, and we merely have $d\xi_s = \dot\xi_s\, ds$. The parameter s is precisely the physical proper time.

We shall use the following smooth coordinates $(\lambda\,,\,\mu\,,\,x\,,\,y\,,\,z) \in \mathbb{R}^5$ on $T^1_+\mathbb{R}^{1,2}$:

$$\dot\xi^0 = \mathrm{ch}\,\lambda\,\mathrm{ch}\,\mu\,;\quad \dot\xi^1 = \mathrm{ch}\,\lambda\,\mathrm{sh}\,\mu\,;\quad \dot\xi^2 = \mathrm{sh}\,\lambda\,;\quad \xi^0 = x\,;\quad \xi^1 = y\,;\quad \xi^2 = z\,. \tag{3}$$

In these coordinates the Dudley diffusion $X_s \equiv (\lambda_s\,,\,\mu_s\,,\,x_s\,,\,y_s\,,\,z_s) \in \mathbb{R}^5$ satisfies the following system of stochastic differential equations (for independent real Brownian motions w,β):

$$d\lambda_s = \sigma\, dw_s + \tfrac{\sigma^2}{2}\,\mathrm{th}\,\lambda_s\, ds\,;\quad d\mu_s = \sigma\,\frac{d\beta_s}{\mathrm{ch}\,\lambda_s}\,; \tag{4}$$

$$dx_s = \mathrm{ch}\,\lambda_s\,\mathrm{ch}\,\mu_s\, ds\,;\quad dy_s = \mathrm{ch}\,\lambda_s\,\mathrm{sh}\,\mu_s\, ds\,;\quad dz_s = \mathrm{sh}\,\lambda_s\, ds\,. \tag{5}$$

The infinitesimal generator $\mathcal{L}$ of (X_s) reads in these coordinates:

$$\mathcal{L} = \tfrac{\sigma^2}{2}\left[\frac{\partial^2}{\partial\lambda^2} + \mathrm{th}\,\lambda\,\frac{\partial}{\partial\lambda} + (\mathrm{ch}\,\lambda)^{-2}\frac{\partial^2}{\partial\mu^2}\right] + \mathrm{ch}\,\lambda\,\mathrm{ch}\,\mu\,\frac{\partial}{\partial x} + \mathrm{ch}\,\lambda\,\mathrm{sh}\,\mu\,\frac{\partial}{\partial y} + \mathrm{sh}\,\lambda\,\frac{\partial}{\partial z}\,\cdot \tag{6}$$

Consider the following smooth vector fields on $\mathbb{R}^5$:

$$V_1 := \sigma\,\frac{\partial}{\partial\lambda}\,, V_2 := \frac{\sigma}{\mathrm{ch}\,\lambda}\,\frac{\partial}{\partial\mu}\,, V_0' := \frac{\partial}{\partial s} + \tfrac{\sigma^2}{2}\,\mathrm{th}\,\lambda\,\frac{\partial}{\partial\lambda} + \mathrm{ch}\,\lambda\,\mathrm{ch}\,\mu\,\frac{\partial}{\partial x} + \mathrm{ch}\,\lambda\,\mathrm{sh}\,\mu\,\frac{\partial}{\partial y} + \mathrm{sh}\,\lambda\,\frac{\partial}{\partial z}\,,$$

and also $V_0 := V_0' - \frac{\partial}{\partial s}\,\cdot$ We have then

$$\tfrac{\partial}{\partial s} + \mathcal{L} = \tfrac12\big(V_1^2 + V_2^2\big) + V_0'\,,\ [V_1,V_0'] = \tfrac{\sigma^3}{2\,\mathrm{ch}^2\lambda}\,\tfrac{\partial}{\partial\lambda} + \sigma\,\mathrm{sh}\,\lambda\,\mathrm{ch}\,\mu\,\tfrac{\partial}{\partial x} + \sigma\,\mathrm{sh}\,\lambda\,\mathrm{sh}\,\mu\,\tfrac{\partial}{\partial y} + \sigma\,\mathrm{ch}\,\lambda\,\tfrac{\partial}{\partial z}\,,$$

$$[V_2\,,V_0'] = \tfrac{\sigma^3\,\mathrm{th}^2\lambda}{2\,\mathrm{ch}\,\lambda}\,\tfrac{\partial}{\partial\lambda} + \sigma\,\mathrm{sh}\,\mu\,\tfrac{\partial}{\partial x} + \sigma\,\mathrm{ch}\,\mu\,\tfrac{\partial}{\partial y}\,,\quad \big[V_2,[V_2,V_0']\big] = \tfrac{\sigma^2}{\mathrm{ch}\,\lambda}\big(\mathrm{ch}\,\mu\,\tfrac{\partial}{\partial x} + \mathrm{sh}\,\mu\,\tfrac{\partial}{\partial y}\big),$$

and then $\Big(V_0', V_1, V_2, [V_1,V_0'], [V_2,V_0'], \big[V_2,[V_2,V_0']\big]\Big)$ has full rank 6 at any point, so that the weak Hörmander condition holds.

Hence by hypoellipticity, $\mathcal{L}$ admits a smooth heat kernel $p_s(X_0;X)$ $(s \in \mathbb{R}_+\,,\ X_0, X \in \mathbb{R}^5)$, with respect to the Liouville measure L, which reads $L(dX) = \mathrm{ch}\,\lambda\, d\lambda\, d\mu\, dx\, dy\, dz$:

$$\mathbb{E}_{X_0}\big[F(X_s)\big] = \int_{T^1_+\mathbb{R}^{1,2}} p_s(X_0;X)\, L(dX) = \int_{\mathbb{R}^5} p_s(X_0;\,\lambda\,,\,\mu\,,\,x\,,\,y\,,\,z)\,\mathrm{ch}\,\lambda\, d\lambda\, d\mu\, dx\, dy\, dz\,.$$

An open question is to estimate $p_s(X_0\,;X)$, for small proper times s.

Up to apply some element of the Poincaré group $\mathcal{P}^3$, we can restrict to $X_0 = (e_0, 0) \equiv (0,0,0,0,0)$. Thus we have to deal with $p_s(0,X) \equiv p_s(X)$, for $X \equiv (\lambda\,,\,\mu\,,\,x\,,\,y\,,\,z) \in \mathbb{R}^5$.

The underlying unperturbed (deterministic) process $X^0 \equiv (\lambda^0\,,\,\mu^0\,,\,x^0\,,\,y^0\,,\,z^0) \in \mathbb{R}^5$ solves:

$$d\lambda^0_s = \tfrac{\sigma^2}{2}\,\mathrm{th}\,\lambda^0_s\,ds\,; \quad \mu^0_s = 0\,; \quad dx^0_s = \mathrm{ch}\,\lambda^0_s\,ds\,; \quad y^0_s = 0\,; \quad dz^0_s = \mathrm{sh}\,\lambda^0_s\,ds\,,$$

and then is merely given by the geodesic $X^0_s \equiv (0\,,\,0\,,\,s\,,\,0\,,\,0)$ (for any proper time s).

Up to change the speed of the canonical Brownian motion (w,β), by considering $(w_{\sigma^2 s}, \beta_{\sigma^2 s})$ instead of (w_s, β_s), we can absorb the speed parameter σ, and then suppose that $\sigma = 1$.

3 A process tangent to the relativistic diffusion (X_s)

The main Theorem 2.1 in [Ca] could apply here (beware however that V_0 is unbounded), yielding a full general Taylor expansion for the diffusion (X_s), in terms of the above vector fields V_1, V_2, V_0, their successive brackets, and of the iterated Stratonovich integrals with respect to (w,β). Indeed, Equations (4),(5) read equivalently:

$$dX_s = V_1(X_s)\,dw_s + V_2(X_s)\,d\beta_s + V_0'(X_s)\,ds\,.$$

In [Ca], the successive remainders corresponding to the truncated Taylor expansion are controlled in probability. In this spirit and also almost surely, the process X_s is approached as follows.

Lemma 3.1 *(i) For any $\varepsilon > 0$, almost surely as proper time $s \searrow 0$ we have:*

$$X_s = \Big(w_s + \tfrac12 \int_0^s w_\tau\,d\tau,\ \beta_s + o(s^{3/2-\varepsilon}),\ s + \tfrac12 \int_0^s \big[\beta_\tau^2 + w_\tau^2\big]d\tau,\ \int_0^s \beta_\tau\,d\tau,\ \int_0^s w_\tau\,d\tau\Big) + o(s^{5/2-\varepsilon}).$$

(ii) Setting $R'_s := (x_s, y_s, z_s) - \Big(s + \tfrac12 \int_0^s \big[\beta_\tau^2 + w_\tau^2\big]d\tau,\ \int_0^s \beta_\tau\,d\tau,\ \int_0^s w_\tau\,d\tau\Big)$ *and*

$R_s := (\lambda_s, \mu_s) - (w_s, \beta_s)$, *there exist $c, \kappa > 0$ such that for any $R > c$ we have:*

$$\lim_{s\searrow 0} \mathbb{P}\Big[\sup_{0\le t\le s} \|R_t\| \ge R\,s^{3/2}\Big] \le e^{-R^\kappa/c} \quad \text{and} \quad \lim_{s\searrow 0} \mathbb{P}\Big[\sup_{0\le t\le s} \|R'_t\| \ge R\,s^{5/2}\Big] \le e^{-R^\kappa/c}.$$

The proof is posponed to Section 7.

Remark 3.2 More precisely, concerning the martingale (μ_s) we have

$$\mu_s = \int_0^s \frac{d\beta_\tau}{\mathrm{ch}\,\lambda_\tau} = \int_0^s \big(1 - \tfrac12\,w_\tau^2 + o(\tau^{2-\varepsilon})\big)d\beta_\tau = \beta_s - \tfrac12 \int_0^s w_\tau^2\,d\beta_\tau + o(s^{5/2-\varepsilon}).$$

But the method used then does not work with the non-quadratic martingale $\int_0^s w_\tau^2\,d\beta_\tau$
$\big($which equals $w_s^2\beta_s - 2\int_0^s \beta_\tau w_\tau dw_\tau - \int_0^s \beta_\tau d\tau = o(s^{3/2-\varepsilon})\big)$.

As a consequence, we shall use a perturbation method, approaching (for small proper time s) the relativistic diffusion X_s by means of the $\mathbb{R}^5$-valued "tangent process" :

$$Y_s := \Big(w_s \,,\ \beta_s \,,\ \tfrac{1}{2}\int_0^s \big[\beta_\tau^2 + w_\tau^2\big] d\tau \,,\ \int_0^s \beta_\tau \, d\tau \,,\ \int_0^s w_\tau \, d\tau \Big) =: \big(w_s \,,\ \beta_s \,,\ A_s \,,\ \zeta_s \,,\ \bar{z}_s \big) \qquad (7)$$

which is not Gaussian, but has its third coordinate A_s in the second Wiener chaos. This actually yields the orthogonal projection of the process (X_s) onto the second Wiener chaos.

Remark 3.3 The fact that the second chaos term is needed in the approximation (without it, the tangent process would clearly not admit any density) makes a significant difference with the situation exhaustively investigated in [DM], where the approaching process is Gaussian. This can no longer be the case in the present setting, though both settings share the feature of being strictly hypoelliptic. A difference between both is the curvature, at the heart of the relativistic realm (even in the present Minkowski-Dudley flat case), due to the mass shell constraint on velocities.

Note that for any fixed proper time $s > 0$ we have:

$$Y_s \overset{law}{\equiv} \Big(\sqrt{s}\, w_1 \,,\ \sqrt{s}\, \beta_1 \,,\ \tfrac{s^2}{2}\int_0^1 \big[\beta_\tau^2 + w_\tau^2\big] d\tau \,,\ \sqrt{s^3}\int_0^1 \beta_\tau \, d\tau \,,\ \sqrt{s^3}\int_0^1 w_\tau \, d\tau \Big). \qquad (8)$$

Denote by $q_s = q_s(w, \beta, x, \zeta, z)$ the density of Y_s with respect to the Lebesgue measure $\Lambda(dY) = dw\, d\beta\, dx\, d\zeta\, dz$ (i.e., *not* the Liouville measure L) on $\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{R}^2$.

By the scaling property (8), it must satisfy :

$$q_s(w, \beta, x, \zeta, z) = \frac{1}{s^6}\ q_1\Big(\frac{w}{\sqrt{s}} \,,\ \frac{\beta}{\sqrt{s}} \,,\ \frac{x}{s^2} \,,\ \frac{\zeta}{\sqrt{s^3}} \,,\ \frac{z}{\sqrt{s^3}} \Big) \,, \qquad (9)$$

and otherwise: $\quad q_s(w, \beta, x, \zeta, z) = q_s(-w, \beta, x, \zeta, -z) = q_s(w, -\beta, x, -\zeta, z)\,.$

4 Fourier-Laplace transform of the tangent process

4.1 Fourier-Laplace transform of the simplified process Z_s

We need information on the density at time s of the tangent process (Y_s) given in (7). By the independence of w, β, it will be enough to consider the density of

$$Z_s := \Big(w_s \,,\ \int_0^s w_\tau \, d\tau \,,\ \int_0^s w_\tau^2 \, d\tau \Big) \overset{law}{\equiv} \Big(\sqrt{s}\, w_1 \,,\ \sqrt{s^3}\int_0^1 w_\tau \, d\tau \,,\ s^2 \int_0^1 w_\tau^2 \, d\tau \Big). \qquad (10)$$

Note that this simplified tangent process Z_s does not have any component beyond the second chaos. Because of the scaling property (10) of Z_s, its density $q_s^0(w, z, x)$ satisfies

$$q_s^0(w, z, x) = \frac{1}{s^4}\ q_1^0\Big(\frac{w}{\sqrt{s}} \,,\ \frac{z}{\sqrt{s^3}} \,,\ \frac{x}{s^2} \Big) \ = q_s^0(-w, -z, x)\,.$$

Of course, the Langevin process $\left(w_s\,,\int_0^s w_\tau\,d\tau\right)$ is Gaussian with covariance

$K_s^0 = \begin{pmatrix} s & s^2/2 \\ s^2/2 & s^3/3 \end{pmatrix}$, so that it has the well-known density

$$(w,z) \longmapsto \frac{\sqrt{3}}{\pi\, s^2}\, e^{-(6z^2-6s\,zw+2s^2w^2)/s^3} = \frac{\sqrt{3}}{\pi\, s^2}\, \exp\left[-\frac{6}{s^3}\Big(z-\frac{s}{2}\,w\Big)^2 - \frac{w^2}{2s}\right]. \tag{11}$$

(In particular, the expected value of $\int_0^s w_\tau\,d\tau$, conditionally on $w_s = w$, equals $sw/2$.) The law of the variable Z_s is not at all that simple, but it is known (see [Y], [CDJR]) that its Fourier-Laplace transform is computable. The following lemma is proved in Section 7.

Lemma 4.1.1 *The law of the variable Z_s of (10) is given by: for any $s \geq 0$ and real r, c, b,*

$$\mathbb{E}_0\left[\exp\left(\sqrt{-1}\,\left[r\,w_s + c\int_0^s w_\tau\,d\tau\right] - \frac{b^2}{2}\int_0^s w_\tau^2\,d\tau\right)\right]$$
$$= \frac{1}{\sqrt{\operatorname{ch}(bs)}}\,\exp\left[-\frac{\operatorname{th}(bs)}{2\,b}\,r^2 - 2\,\frac{\operatorname{sh}^2(bs/2)}{b^2\operatorname{ch}(bs)}\,rc - \frac{bs-\operatorname{th}(bs)}{2\,b^3}\,c^2\right].$$

Of course, for $b=0$ we recover the Fourier transform of (11), namely $e^{-\left(r^2+s\,rc+\frac{s^2}{3}c^2\right)s/2}$.

Proposition 4.1.2 *The x-Laplace transform of the variable Z_1 of (10) is given by: for any real w, z, b,*

$$\int_0^\infty e^{-\frac{b^2}{2}x}\,q_1^0(w,z,x)\,dx = \frac{b^2\,\exp\left[-\frac{\left[b\,z-\operatorname{th}(b/2)\,w\right]^2+\coth b\,\left[b-2\operatorname{th}(b/2)\right]w^2}{2\left[1-(2/b)\operatorname{th}(b/2)\right]}\right]}{2\pi\sqrt{\left[b-2\operatorname{th}(b/2)\right]\operatorname{sh} b}}$$

$$= \frac{b^2}{2\pi\sqrt{\left[b-2\operatorname{th}(b/2)\right]\operatorname{sh} b}} \times \exp\left[\frac{b^2}{8}\times\frac{(w-2z)^2}{1-\frac{b}{2}\coth(\frac{b}{2})} - \frac{b^2}{2}\,z^2 - \frac{b}{4}\coth(\tfrac{b}{2})\,w^2\right].$$

This is of course consistent with (11), via $b\to 0$; and integrating with respect to $dw\,dz$, we recover $\int_0^\infty e^{-\frac{b^2}{2}x}\left[\iint q_1^0(w,z,x)\,dw\,dz\right]dx = (\operatorname{ch} b)^{-1/2}$, as it must be.

Proof We invert the Fourier transform in Lemma 4.1.1 by Plancherel's Formula :

$$\int_0^\infty e^{-\frac{b^2}{2}x}\,q_1^0(w,z,x)\,dx$$
$$= \frac{1}{4\pi^2\sqrt{\operatorname{ch} b}}\int_{\mathbb{R}^2} e^{-\sqrt{-1}\,[w\,r+z\,c]}\,\exp\left[-\frac{\operatorname{th} b}{2\,b}\,r^2 - 2\,\frac{\operatorname{sh}^2(b/2)}{b^2\operatorname{ch} b}\,rc - \frac{b-\operatorname{th} b}{2\,b^3}\,c^2\right]dr\,dc$$
$$= \frac{1}{4\pi^2\sqrt{\operatorname{ch} b}}\int_{\mathbb{R}^2} e^{-\sqrt{-1}\,[w\,r+z\,c]}\,\exp\left[-\frac{\operatorname{th} b}{2\,b}\left(r+\frac{\operatorname{th}(b/2)}{b}\,c\right)^2 - \frac{b-2\operatorname{th}(b/2)}{2\,b^3}\,c^2\right]dr\,dc$$

$$= \frac{1}{4\pi^2\sqrt{\operatorname{ch} b}} \int_{\mathbb{R}^2} e^{-\sqrt{-1}\,\left[w\,r + \left(z - \frac{\operatorname{th}(b/2)}{b}\,w\right)c\right]} \,\exp\!\left[-\frac{\operatorname{th} b}{2\,b}\,r^2 - \frac{b - 2\operatorname{th}(b/2)}{2\,b^3}\,c^2\right] dr\,dc$$

$$= \sqrt{\frac{b}{\operatorname{sh} b}} \times \frac{e^{-\frac{b\,w^2}{2\operatorname{th} b}}}{2\pi} \int_{\mathbb{R}} e^{-\sqrt{-1}\,\left(z - \frac{\operatorname{th}(b/2)}{b}\,w\right)c} \,\exp\!\left[-\frac{b - 2\operatorname{th}(b/2)}{2\,b^3}\,c^2\right] \frac{dc}{\sqrt{2\pi}}$$

$$= \sqrt{\frac{b}{\operatorname{sh} b}} \times \frac{e^{-\frac{b\,w^2}{2\operatorname{th} b}}}{2\pi} \times \sqrt{\frac{b^3}{b - 2\operatorname{th}(b/2)}} \times \exp\!\left[-\frac{b^3}{2\big[b - 2\operatorname{th}(b/2)\big]}\left(z - \frac{\operatorname{th}(b/2)}{b}\,w\right)^2\right]$$

$$= \frac{b^2}{2\pi\sqrt{\big[b - 2\operatorname{th}(b/2)\big]\operatorname{sh} b}} \times \exp\!\left[-\frac{\big[bz - \operatorname{th}(b/2)\,w\big]^2 + \coth b\,\big[b - 2\operatorname{th}(b/2)\big]w^2}{2\,\big[1 - (2/b)\operatorname{th}(b/2)\big]}\right]. \ \diamond$$

4.2 Fourier-Laplace transform of the tangent process Y_s

We use Lemma 4.1.1 to express this Fourier-Laplace transform.

Lemma 4.2.1 *The law of the variable Y_s of (7) is given by: for any $s \geq 0$ and real r, ϱ, b, γ, c,*

$$\mathbb{E}_0\Big[\exp\Big(\sqrt{-1}\,\left[r\,w_s + \varrho\,\beta_s + \gamma\,\zeta_s + c\,\bar{z}_s\right] - b^2 A_s\Big)\Big]$$

$$= \frac{1}{\operatorname{ch}(bs)}\,\exp\!\left[-\frac{\operatorname{th}(bs)}{2\,b}\,(r^2 + \varrho^2) - 2\,\frac{\operatorname{sh}^2(bs/2)}{b^2\operatorname{ch}(bs)}\,(r\,c + \varrho\,\gamma) - \frac{bs - \operatorname{th}(bs)}{2\,b^3}\,(c^2 + \gamma^2)\right].$$

In particular, the law of A_s is given by $\mathbb{E}_0\big[\exp(-b^2 A_s)\big] = 1/\operatorname{ch}(bs)$.

Proof This follows directly from Lemma 4.1.1, by independence of w and β. $\diamond$

Proposition 4.2.2 *The x-Laplace transform of the variable Y_1 of (7) is given by: for any real w, β, z, ζ, b,*

$$\int_0^\infty e^{-b^2 x}\, q_1(w, \beta, x, \zeta, z)\, dx$$

$$= \frac{b^4\,\exp\!\left[\frac{b^2}{8} \times \frac{(w-2z)^2 + (\beta-2\zeta)^2}{1 - \frac{b}{2}\coth(\frac{b}{2})} - \frac{b^2}{2}(z^2 + \zeta^2) - \frac{b}{4}\coth(\frac{b}{2})(w^2 + \beta^2)\right]}{8\pi^2\,\big[b\operatorname{ch}(\frac{b}{2}) - 2\operatorname{sh}(\frac{b}{2})\big]\operatorname{sh}(\frac{b}{2})} =: \Psi_{w,\beta,\zeta,z}(b)\,. \qquad (12)$$

In particular, $\Psi_{w,\beta,\zeta,z}(0) = \frac{3}{\pi^2}\exp\!\Big[-\frac{w^2+\beta^2}{2} - 6\big(z - w/2\big)^2 - 6\big(\zeta - \beta/2\big)^2\Big]$ is the marginal density of $\big(w_1\,,\,\beta_1\,,\,\zeta_1\,,\,\bar{z}_1\big)$.

Proof As Lemma 4.2.1 follows from Lemma 4.1.1, this follows merely from Proposition 4.1.2 by independence of w and β. Indeed, for any test functions f, g on $\mathbb{R}^2$ we have:

$$\int_{\mathbb{R}^4} f(w,z)\, g(\beta,\zeta)\left[\int_0^\infty e^{-b^2 x}\, q_1(w,\beta,x,\zeta,z)\,dx\right] dw\,dz\,d\beta\,d\zeta$$

$$= \mathbb{E}\Big[f(w_1,\bar z_1)\, g(\beta_1,\zeta_1)\, e^{-\frac{b^2}{2}\left[\int_0^1 w_\tau^2\,d\tau + \int_0^1 \beta_\tau^2\,d\tau\right]}\Big]$$

$$= \mathbb{E}\Big[f(w_1,\bar z_1)\, e^{-\frac{b^2}{2}\int_0^1 w_\tau^2\,d\tau}\Big] \times \mathbb{E}\Big[g(\beta_1,\zeta_1)\, e^{-\frac{b^2}{2}\int_0^1 \beta_\tau^2\,d\tau}\Big]$$

$$= \int_{\mathbb{R}^2} f(w,z) \int_0^\infty e^{-\frac{b^2}{2}x}\, q_1^0(w,z,x)\,dx\ dw\,dz \times \int_{\mathbb{R}^2} g(\beta,\zeta) \int_0^\infty e^{-\frac{b^2}{2}x}\, q_1^0(\beta,\zeta,x)\,dx\ d\beta\,d\zeta$$

$$= \int_{\mathbb{R}^4} f(w,z)\, g(\beta,\zeta) \left[\int_0^\infty e^{-\frac{b^2}{2}x}\, q_1^0(w,z,x)\,dx \times \int_0^\infty e^{-\frac{b^2}{2}x}\, q_1^0(\beta,\zeta,x)\,dx\right] dw\,dz\,d\beta\,d\zeta\,,$$

and the claim follows directly from Proposition 4.1.2. $\diamond$

Lemma 4.2.3 *All solutions $z \in \mathbb{C}^*$ of the equation $z = \operatorname{th} z$ belong to the imaginary axis, and form a sequence $\mathcal{R} = \{\pm\sqrt{-1}\, y_n \,|\, n \in \mathbb{N}\}$, with $\frac{17\pi}{12} < y_0 < y_1 < \ldots < y_n \nearrow \infty$.*

<u>Proof</u> Writing $z = x + \sqrt{-1}\, y \in \mathbb{C}^*$, we have $z = \operatorname{th} z \Leftrightarrow e^{2z} = \frac{1+z}{1-z}$ and then equivalently

$$\cos(2y) = \tfrac{1-x^2-y^2}{(1-x)^2+y^2}\, e^{-2x} \quad \text{ and } \quad \sin(2y) = \tfrac{2\,y}{(1-x)^2+y^2}\, e^{-2x}\ ,$$

whence

$$e^{4x}\big((1-x)^2+y^2\big)^2 = 4y^2 + (1-x^2-y^2)^2\,.$$

The latter is equivalent either to $x=0$, or to $y^2 = -(1-x)^2$ (which is excluded), or to $y^2 = \frac{(1+x)^2 - e^{4x}(1-x)^2}{e^{4x}-1}\cdot$ Then using this last value of y^2, by the above we must also have $\sin(2y) = \frac{y}{x}\operatorname{sh}(2x)$, which is impossible since for any $x, y \in \mathbb{R}^*$ we have $\frac{\sin(2y)}{y} < 2 < \frac{\operatorname{sh}(2x)}{x}$, and clearly z cannot be real. Hence we are left with $z = \pm\sqrt{-1}\, y$, with $y > 0$ and then $y = \operatorname{tg} y$. The claim follows, with moreover $\big((n+3/2)\pi - y_n\big) \searrow 0$, and $\frac{3\pi}{2} > y_0 > \frac{17\pi}{12}$ since $\operatorname{tg}\frac{17\pi}{12} = \operatorname{cotg}\frac{\pi}{12} = 2+\sqrt{3} < \frac{17\pi}{12}\cdot$ $\diamond$

The Laplace transform $\Psi_{w,\beta,\zeta,z}(b)$ in Proposition 4.2.2 is a meromorphic function of $b \in \mathbb{C}$, with singularities at the points of $\sqrt{-1}\ 2\pi\,\mathbb{Z}^*$ and at the non-null zeros of $\big[b - 2\operatorname{th}(b/2)\big]$, that is to say at the points of $2\mathcal{R}$ (according to Lemma 4.2.3).

4.3 Complement : Density $\alpha_s(x)$ of the variable A_s

Denote by $\alpha_s = \alpha_s(x)$ the density of the variable A_s, so that for any $s > 0$ we have

$$\alpha_s(x) = \int q_s(w,\beta,x,\zeta,z)\,dw\,d\beta\,d\zeta\,dz \ =\ \frac{1}{s^2}\,\alpha_1\Big(\frac{x}{s^2}\Big), \text{ by (9).}$$

Lemma 4.3.1 *The density α_1 is smooth and bounded (with bounded derivatives), and we have*

$$\alpha_1(x) = \frac{4}{\pi}\int_0^\infty \frac{\cos\!\big(2x\,y^2 - y\big)}{\operatorname{sh}^2 y + \cos^2 y}\; y \operatorname{sh} y\,dy + \frac{4}{\pi}\int_0^\infty \frac{\cos y\,\cos(2x\,y^2)}{\operatorname{sh}^2 y + \cos^2 y}\; e^{-y}\, y\,dy\,.$$

Proof According to Lemma 4.2.1, for any $\eta > 0$ we have: $\alpha_1(x) =$

$$\frac{e^{\eta x}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\sqrt{-1}\,x\,y}}{\mathrm{ch}\Big(\sqrt{\eta+\sqrt{-1}\,y}\Big)}\,dy \;=\; \frac{e^{\eta x}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\sqrt{-1}\,x\,y}}{\mathrm{ch}\Bigg(\sqrt{\frac{\sqrt{y^2+\eta^2}+\eta}{2}}+\sqrt{-1}\,\mathrm{sgn}(y)\sqrt{\frac{\sqrt{y^2+\eta^2}-\eta}{2}}\,\Bigg)}\,dy\,.$$

Expanding and letting $\eta \searrow 0$ we obtain:

$$\alpha_1(x) = \frac{4}{\pi}\int_0^{\infty}\frac{\mathrm{ch}\,y\,\cos y\,\cos(2x\,y^2)+\mathrm{sh}\,y\,\sin y\,\sin(2x\,y^2)}{\mathrm{sh}^2y+\cos^2 y}\,y\,dy$$

$$= \frac{4}{\pi}\int_0^{\infty}\frac{\cos\big(2x\,y^2-y\big)}{\mathrm{sh}^2y+\cos^2 y}\,y\,\mathrm{sh}\,y\,dy + \frac{4}{\pi}\int_0^{\infty}\frac{\cos y\,\cos(2x\,y^2)}{\mathrm{sh}^2y+\cos^2 y}\,e^{-y}\,y\,dy\,.$$

As a consequence, α_1 is smooth, and bounded by $\frac{4}{\pi}\int_0^{\infty}\frac{y\,\mathrm{ch}\,y\,dy}{\mathrm{sh}^2y+\cos^2 y}\cdot$ $\diamond$

Remark 4.3.2 *We have* $\mathrm{sh}^2z+\cos^2z=0 \Leftrightarrow \mathrm{ch}^2z=\sin^2z \Leftrightarrow z\in(1\pm\sqrt{-1}\,)\frac{\pi}{4}(1+2\mathbb{Z})$,

$$i.e.,\quad |z|\in\tfrac{\pi(1+2\mathbb{N})}{2\sqrt{2}}\,,\quad \mathrm{Arg}\,z\;\equiv\;\tfrac{\pi}{4}\ \ modulo\ \ \tfrac{\pi}{2}\,\cdot$$

Proof There is clearly no real solution, since $\mathrm{ch}\,x > 1 \geq \sin x$ for any $x\in\mathbb{R}^*$. Consider then $z = (\sqrt{-1}\,+\alpha)\,x$, with $x,\alpha\in\mathbb{R}$. Up to change z into $-z$, we only have to consider the equation $\mathrm{ch}\,z=\sin z$. Then

$$\mathrm{ch}\,z=\sin z \iff \mathrm{ch}(\alpha x)\cos x = \mathrm{ch}\,x\,\sin(\alpha x)\ \text{ and }\ \mathrm{sh}(\alpha x)\sin x = \mathrm{sh}\,x\,\cos(\alpha x)$$

$$\Rightarrow \mathrm{ch}^2(\alpha x)\cos^2 x+\mathrm{sh}^2(\alpha x)\sin^2 x = \mathrm{ch}^2x\,\mathrm{sh}^2x \iff \big(\mathrm{ch}^2x-\cos^2x\big)\big(\mathrm{sh}^2(\alpha x)-\mathrm{sh}^2x\big)=0\,.$$

As $x=0$ is not a solution, the only possibility is $\mathrm{sh}(\alpha x)=\pm\,\mathrm{sh}\,x$, whence $\alpha=\pm1$, and then $\cos x=\alpha\,\sin x$. This yields the claim. $\diamond$

Proposition 4.3.3 *We have* $\alpha_1(x) = 2\pi\sum_{n\in\mathbb{N}}(-1)^n\big(n+\frac{1}{2}\big)\,e^{-\left(n+\frac{1}{2}\right)^2\pi^2 x}$, *so that*

$$\pi\,e^{-\pi^2x/4}\big(1-3\,e^{-2\pi^2x}\big)\leq\alpha_1(x)\leq\pi\,e^{-\pi^2x/4}\quad \textit{for any } x\geq\pi^{-2},\ \textit{and then}$$

$$\alpha_1(x)\;=\;\pi\,e^{-\pi^2x/4}\,\big(1-\mathcal{O}(e^{-2\pi^2x})\big)\ \ \textit{as } x\to\infty\,.\ \textit{Moreover, } \alpha_1\ \textit{decreases on } [3\pi^{-2},\infty[\,.$$

Proof This results from Lemma 4.2.1 and $\big($[BPY], Table 1 continued and Section 3.3, in particular Formula (3.11), with $C_1\equiv 2A_1$, got merely by expanding $1/(\mathrm{ch}\,t)\big)$. For any $\pi^2x>1$, this alternate series has decreasing generic term $\big(n+\frac{1}{2}\big)\,e^{-\left(n+\frac{1}{2}\right)^2\pi^2x}$, whence the estimate. Finally, the same holds for the series $\big(n+\frac{1}{2}\big)^3\,e^{-\left(n+\frac{1}{2}\right)^2\pi^2x}$ yielding $\alpha_1'(x)$, thereby garanteing $\alpha_1'(x)<0$, as soon as $\pi^2x>3$. $\diamond$

Lemma 4.2.1 and Proposition 4.3.3 at once entail the following.

Corollary 4.3.4 $\lambda \longmapsto \int_0^\infty e^{\lambda x}\,\alpha_1(x)\,dx$ *defines an analytic function on* $\Re(\lambda) < \pi^2/4$, *equal to* $\frac{1}{\cos\sqrt{\lambda}}$ *for* $0 \le \lambda < \frac{\pi^2}{4}$. *Moreover, we have* $\alpha_s(x) = \frac{\pi}{s^2}\, e^{-\pi^2 x/(4s^2)}\big(1 - \mathcal{O}(e^{-2\pi^2 x/s^2})\big)$ *(as* $\frac{x}{s^2} \to \infty$*).*

Proof By Lemma 4.2.1 the set of $b \in \mathbb{C}$ such that $\int_0^\infty e^{-b^2 x}\,\alpha_1(x)\,dx = \frac{1}{\operatorname{ch} b}$ contains $\mathbb{R}$. By Proposition 4.3.3 $b \mapsto \int_0^\infty e^{-b^2 x}\,\alpha_1(x)\,dx$ is analytic on $\big\{|\Im(b)| < \pi/2\big\}$, and so is $b \mapsto \frac{1}{\operatorname{ch} b}$ too. Hence we have $\int_0^\infty e^{b^2 x}\,\alpha_1(x)\,dx = \frac{1}{\cos b}$ for $|\Re(b)| < \pi/2$. $\diamond$

Remark 4.3.5 1) The law of $2\,A_1$ is that of $\inf\big\{\tau > 0 \,\big|\, |\beta_\tau| = 1\big\}$ and also that of $\Big(\max\big\{|\beta_\tau| \,\big|\, 0 \le \tau \le 1\big\}\Big)^{-2}$ (See [BPY], Table 2, which also exhibits two other random variables having the same law as A_1). 2) Note that for $\Re(\lambda) < \pi^2/4$ (letting $\sqrt{\cdot} > 0$ on $\mathbb{R}_+^*$) :

$$\frac{1}{\cos\sqrt{\lambda}} = \int_0^\infty\!\!\int e^{\lambda x}\, q_1(w,\beta,x,\zeta,z)\,dw\,d\beta\,d\zeta\,dz\,dx = \int \Psi_{w,\beta,\zeta,z}\big(\sqrt{-\lambda}\big)\,dw\,d\beta\,d\zeta\,dz\,.$$

5 Integral expression of the density $q_s(w,\beta,x,\zeta,z)$

Let

$$\tilde q_s \equiv \tilde q_s(w,\beta,\zeta,z) := \frac{3}{\pi^2 s^4}\,\exp\bigg[-\frac{3\big[(sw-2z)^2 + (s\beta-2\zeta)^2\big]}{2\,s^3} - \frac{w^2+\beta^2}{2\,s}\bigg] \tag{13}$$

denote the marginal Gaussian density of $\tilde Y_s := \Big(w_s\,,\ \beta_s\,,\int_0^s \beta_\tau\,d\tau\,,\int_0^s w_\tau\,d\tau\Big)$ (according to (11) or Proposition 4.2.2)).

Notation Consider the function $\Phi(\lambda) \equiv \Phi_{w,\beta,\zeta,z}(\lambda) := \Psi_{w,\beta,\zeta,z}\big(\sqrt{-\lambda}\big)$, derived from Proposition 4.2.2. We systematically use the usual determination of the complex square root, cutting $\mathbb{C}$ along the negative real semi-axis and letting $\sqrt{\cdot} > 0$ on $\mathbb{R}_+^*$. By the expression (12), we have:

$$\Phi_{w,\beta,\zeta,z}(\lambda) = \frac{\lambda^2\,\exp\Big[\frac{\lambda}{8}\times\frac{(w-2z)^2+(\beta-2\zeta)^2}{\frac{\sqrt{\lambda}}{2}\operatorname{cotg}(\frac{\sqrt{\lambda}}{2})-1} + \frac{\lambda}{2}(z^2+\zeta^2) - (w^2+\beta^2)\frac{\sqrt{\lambda}}{4}\operatorname{cotg}(\frac{\sqrt{\lambda}}{2})\Big]}{8\pi^2\,\big[2\sin(\frac{\sqrt{\lambda}}{2}) - \sqrt{\lambda}\cos(\frac{\sqrt{\lambda}}{2})\big]\sin(\frac{\sqrt{\lambda}}{2})}\,. \tag{14}$$

Lemma 5.1 *The function* $\lambda \mapsto \Phi_{w,\beta,\zeta,z}(\lambda)$ *is analytic for* $\Re(\lambda) < 4\pi^2$.

Proof Note that the functions $\sin(\frac{\sqrt{\lambda}}{2})\Big/\frac{\sqrt{\lambda}}{2}$ and $\cos(\frac{\sqrt{\lambda}}{2})$ are plainly analytically continued for any $\lambda \in \mathbb{C}$, and that by the expression (14), $\Phi(\lambda)$ is analytically continued at $\lambda = 0$, and analytic at any $\lambda \in \mathbb{C}^*$ such that $\sin(\frac{\sqrt{\lambda}}{2}) \neq 0$ and $\operatorname{tg}(\frac{\sqrt{\lambda}}{2}) \neq \frac{\sqrt{\lambda}}{2}$, hence, according to Lemma 4.2.3, at those $\lambda \in \mathbb{C}$ not belonging to the sequence $\{4\pi^2(n+1)^2, 4\,y_n^2 \,|\, n \in \mathbb{N}\} \subset [4\pi^2, \infty[$. This shows the analyticity for $\Re(\lambda) < 4\pi^2$. $\diamond$

The following is proved in Section 7.

Proposition 5.2 *For any* (w,β,ζ,z) *we have*

$$\Phi_{w,\beta,\zeta,z}(\lambda) = \int_0^\infty e^{\lambda x}\, q_1(w,\beta,x,\zeta,z)\,dx\,, \qquad \text{for } \Re(\lambda) < 4\pi^2, \text{ and}$$

$$e^{r\,x}\, q_1(w,\beta,x,\zeta,z) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\,t\,x}\,\Phi(r+\sqrt{-1}\;t)\,\frac{dt}{2\pi}\,, \qquad \text{for } x>0 \text{ and } r<4\pi^2. \tag{15}$$

By scaling and using (15), for any $s\,,\,x\,>0\,,\,r<4\pi^2$ and $(w,\beta,\zeta,z)\in\mathbb{R}^4$ we have:

$$q_s(w,\beta,x,\zeta,z) \;=\; \frac{1}{s^6}\; q_1\!\left(\frac{w}{\sqrt{s}}\,,\frac{\beta}{\sqrt{s}}\,,\frac{x}{s^2}\,,\frac{\zeta}{\sqrt{s^3}}\,,\frac{z}{\sqrt{s^3}}\right)$$

$$=\; \frac{e^{-r\,x/s^2}}{2\pi\,s^6}\int_{-\infty}^{\infty} e^{-\sqrt{-1}\;x\,y/s^2}\;\Phi_{\left(\frac{w}{\sqrt{s}}\,,\frac{\beta}{\sqrt{s}}\,,\frac{\zeta}{\sqrt{s^3}}\,,\frac{z}{\sqrt{s^3}}\right)}(r+\sqrt{-1}\;y)\,dy\,.$$

Taking merely $r=0$, and setting $\quad \widetilde{\Psi}_s := e^{-\sqrt{-1}\;x\,y/s^2}\;\Phi_{\left(\frac{w}{\sqrt{s}},\frac{\beta}{\sqrt{s}},\frac{\zeta}{\sqrt{s^3}},\frac{z}{\sqrt{s^3}}\right)}(\sqrt{-1}\;y)$

for convenience, for any $s,x\,>0$ and $(w,\beta,\zeta,z)\in\mathbb{R}^4$ we have:

$$q_s(w,\beta,x,\zeta,z) \;=\; \frac{1}{2\pi\,s^6}\int_{-\infty}^{\infty}\widetilde{\Psi}_s(y)\,dy \;=\; \frac{1}{2\pi\,s^6}\int_0^{\infty}\big(\widetilde{\Psi}_s(y)+\widetilde{\Psi}_s(-y)\big)\,dy\,. \tag{16}$$

Now according to (14) we have: $\quad \Phi_{\left(\frac{w}{\sqrt{s}},\frac{\beta}{\sqrt{s}},\frac{\zeta}{\sqrt{s^3}},\frac{z}{\sqrt{s^3}}\right)}(\sqrt{-1}\;y) \;=$

$$\frac{-y^2\,e^{B'_s\sqrt{-1}\;y\,s^{-2}-\frac{\sqrt{\sqrt{-1}\;y}}{4s}\operatorname{cotg}\left(\frac{\sqrt{\sqrt{-1}\;y}}{2}\right)(w^2+\beta^2)}}{8\pi^2\big[1-\cos(\sqrt{\sqrt{-1}\;y}\,)-(\sqrt{\sqrt{-1}\;y}/2)\sin(\sqrt{\sqrt{-1}\;y}\,)\big]}\;\exp\left[\frac{B_s^2\sqrt{-1}\;y\,s^{-3}}{1-\frac{2}{\sqrt{\sqrt{-1}\;y}}\operatorname{tg}(\frac{\sqrt{\sqrt{-1}\;y}}{2})}\right],$$

in which we have set

$$B_s^2 := \frac{(sw-2z)^2+(s\beta-2\zeta)^2}{8} \quad \text{ and } \quad B'_s := \frac{4wz+4\beta\zeta-s(w^2+\beta^2)}{8}\,. \tag{17}$$

Let us now write out a more tractable expression of $\widetilde{\Psi}_s$ introduced above.

First, for any real y,θ, setting $\xi := \sqrt{\frac{|y|}{2}}$, we successively have:

$$\sqrt{\sqrt{-1}\;y} = (1+\operatorname{sgn}(y)\sqrt{-1}\,)\,\xi\,; \qquad \frac{2}{\sqrt{\sqrt{-1}\;y}} = (1-\operatorname{sgn}(y)\sqrt{-1}\,)/\xi\,;$$

$$\operatorname{tg}\big((1+\sqrt{-1}\,)\,\theta\big) = \frac{\sin(2\theta)+\sqrt{-1}\;\operatorname{sh}(2\theta)}{\operatorname{ch}(2\theta)+\cos(2\theta)}\,;\; \operatorname{cotg}\big((1+\sqrt{-1}\,)\,\theta\big) = \frac{\sin(2\theta)-\sqrt{-1}\;\operatorname{sh}(2\theta)}{\operatorname{ch}(2\theta)-\cos(2\theta)}\,;$$

$$\operatorname{tg}\Big(\frac{\sqrt{\sqrt{-1}\;y}}{2}\Big) = \frac{\sin\xi+\operatorname{sgn}(y)\sqrt{-1}\;\operatorname{sh}\xi}{\operatorname{ch}\xi+\cos\xi}\,;\; \operatorname{cotg}\Big(\frac{\sqrt{\sqrt{-1}\;y}}{2}\Big) = \frac{\sin\xi-\operatorname{sgn}(y)\sqrt{-1}\;\operatorname{sh}\xi}{\operatorname{ch}\xi-\cos\xi}\,;$$

$$\frac{2}{\sqrt{\sqrt{-1}\,y}}\,\mathrm{tg}\Big(\frac{\sqrt{\sqrt{-1}\,y}}{2}\Big) = \frac{(\mathrm{sh}\,\xi+\sin\xi)+\mathrm{sgn}(y)\sqrt{-1}\;(\mathrm{sh}\,\xi-\sin\xi)}{\xi\,(\mathrm{ch}\,\xi+\cos\xi)}\;;$$

$$\frac{1}{1-\frac{2}{\sqrt{\sqrt{-1}\,y}}\,\mathrm{tg}\Big(\frac{\sqrt{\sqrt{-1}\,y}}{2}\Big)} = \frac{\xi\,(\mathrm{ch}\,\xi+\cos\xi)-(\mathrm{sh}\,\xi+\sin\xi)+\mathrm{sgn}(y)\sqrt{-1}\,(\mathrm{sh}\,\xi-\sin\xi)}{\xi\,(\mathrm{ch}\,\xi+\cos\xi)-2(\mathrm{sh}\,\xi+\sin\xi)+2\xi^{-1}(\mathrm{ch}\,\xi-\cos\xi)}\;;$$

$$\frac{\sqrt{-1}\;y}{1-\frac{2}{\sqrt{\sqrt{-1}\,y}}\,\mathrm{tg}\Big(\frac{\sqrt{\sqrt{-1}\,y}}{2}\Big)} = \frac{y\,\xi\left[\sqrt{-1}\,\left(\xi-\frac{\mathrm{sh}\,\xi+\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)-\mathrm{sgn}(y)\left(\frac{\mathrm{sh}\,\xi-\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)\right]}{\left(\xi-\frac{\mathrm{sh}\,\xi+\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)^2+\left(\frac{\mathrm{sh}\,\xi-\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)^2} =: U(y)\,;$$

$$\frac{\sqrt{\sqrt{-1}\;y}}{2}\,\mathrm{cotg}\Big(\frac{\sqrt{\sqrt{-1}\;y}}{2}\Big) = \frac{(\mathrm{sh}\,\xi+\sin\xi)-\mathrm{sgn}(y)\sqrt{-1}\;(\mathrm{sh}\,\xi-\sin\xi)}{2\,\xi^{-1}\,(\mathrm{ch}\,\xi-\cos\xi)} =: V(y)\,;$$

$$\cos\Big(\sqrt{\sqrt{-1}\;y}\Big) = \mathrm{ch}\,\xi\;\cos\xi-\mathrm{sgn}(y)\sqrt{-1}\;\mathrm{sh}\,\xi\;\sin\xi\;;$$

$$\sin\Big(\sqrt{\sqrt{-1}\;y}\Big) = \mathrm{ch}\,\xi\;\sin\xi+\mathrm{sgn}(y)\sqrt{-1}\;\mathrm{sh}\,\xi\;\cos\xi\;;$$

$$1-\cos\Big(\sqrt{\sqrt{-1}\;y}\Big)-\Big(\sqrt{\sqrt{-1}\;y}/2\Big)\sin\Big(\sqrt{\sqrt{-1}\;y}\Big) =$$

$$1-\mathrm{ch}\,\xi\cos\xi-\tfrac{\xi}{2}(\mathrm{ch}\,\xi\sin\xi-\mathrm{sh}\,\xi\cos\xi)+\mathrm{sgn}(y)\sqrt{-1}\;\Big[\mathrm{sh}\,\xi\sin\xi-\tfrac{\xi}{2}(\mathrm{ch}\,\xi\sin\xi+\mathrm{sh}\,\xi\cos\xi)\Big].$$

Set

$$F(y) := \frac{-1}{1-\cos\left(\sqrt{\sqrt{-1}\;y}\right)-\left(\sqrt{\sqrt{-1}\;y}/2\right)\sin\left(\sqrt{\sqrt{-1}\;y}\right)} \;=\; F_r(\xi)+\mathrm{sgn}(y)\sqrt{-1}\;F_i(\xi)\,,$$

with

$$F_r(\xi) := \frac{\mathrm{ch}\,\xi\cos\xi+\frac{\xi}{2}(\mathrm{ch}\,\xi\sin\xi-\mathrm{sh}\,\xi\cos\xi)-1}{(\mathrm{ch}\,\xi-\cos\xi)\big[(\mathrm{ch}\,\xi-\cos\xi)-\xi\,(\mathrm{sh}\,\xi+\sin\xi)+\frac{\xi^2}{2}(\mathrm{ch}\,\xi+\cos\xi)\big]}\,,$$

and

$$F_i(\xi) := \frac{\mathrm{sh}\,\xi\sin\xi-\frac{\xi}{2}(\mathrm{ch}\,\xi\sin\xi+\mathrm{sh}\,\xi\cos\xi)}{(\mathrm{ch}\,\xi-\cos\xi)\big[(\mathrm{ch}\,\xi-\cos\xi)-\xi\,(\mathrm{sh}\,\xi+\sin\xi)+\frac{\xi^2}{2}(\mathrm{ch}\,\xi+\cos\xi)\big]}\,.$$

Hence

$$\widetilde{\Psi}_s \equiv \widetilde{\Psi}_s(y) \;=\; \frac{y^2\,F(y)}{8\pi^2}\;\exp\!\left[\sqrt{-1}\;\frac{B_s'-x}{s^2}\,y-\frac{w^2+\beta^2}{2s}\,V(y)+\frac{B_s^2}{s^3}\,U(y)\right] \qquad (18)$$

$$= \frac{y^2\,F(y)}{8\pi^2}\;\exp\!\left[\sqrt{-1}\,\left(\frac{B_s^2}{s^3}\,U_i(\xi)+\frac{B_s'-x}{s^2}+\frac{w^2+\beta^2}{2s}\,V_i(\xi)\right)y\;-\frac{B_s^2}{s^3}\,U_r(\xi)-\frac{w^2+\beta^2}{2s}\,V_r(\xi)\right],$$

with

$$U_i(\xi) := \frac{\xi\left(\xi-\frac{\mathrm{sh}\,\xi+\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)}{\left(\xi-\frac{\mathrm{sh}\,\xi+\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)^2+\left(\frac{\mathrm{sh}\,\xi-\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)^2}\;;\quad U_r(\xi) := \frac{2\,\xi^3\left(\frac{\mathrm{sh}\,\xi-\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)}{\left(\xi-\frac{\mathrm{sh}\,\xi+\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)^2+\left(\frac{\mathrm{sh}\,\xi-\sin\xi}{\mathrm{ch}\,\xi+\cos\xi}\right)^2}\;;$$

$$V_i(\xi) := \frac{\operatorname{sh}\xi - \sin\xi}{4\,\xi\,(\operatorname{ch}\xi - \cos\xi)}\ ; \quad V_r(\xi) := \frac{\xi\,(\operatorname{sh}\xi + \sin\xi)}{2\,(\operatorname{ch}\xi - \cos\xi)}\ .$$

Using these auxiliary functions, (16) reads:

$$q_s(w,\beta,x,\zeta,z)$$

$$= \frac{1}{8\pi^3 s^6}\int_0^\infty \Re\left[F(y)\, e^{\sqrt{-1}\left[\frac{B_s^2}{s^3}\,U_i(\xi)+\frac{B_s'-x}{s^2}+\frac{w^2+\beta^2}{2s}\,V_i(\xi)\right]y}\right] e^{-\frac{B_s^2}{s^3}\,U_r(\xi)-\frac{w^2+\beta^2}{2s}\,V_r(\xi)}\, y^2\,dy$$

$$= \frac{\tilde q_s(w,\beta,\zeta,z)}{(3\pi/2)\,s^2}\int_0^\infty \Re\left[F(y)\, e^{\sqrt{-1}\left[\frac{B_s^2}{s^3}\,U_i(\xi)+\frac{B_s'-x}{s^2}+\frac{w^2+\beta^2}{2s}\,V_i(\xi)\right]2\xi^2}\right] e^{-\frac{B_s^2}{s^3}[U_r(\xi)-12]-\frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\,\xi^5 d\xi$$

$$= \frac{\tilde q_s(w,\beta,\zeta,z)}{3\pi\, s^2}\int_0^\infty G_s(\xi)\, e^{-\frac{B_s^2}{s^3}[U_r(\xi)-12]-\frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\,\xi^5\,d\xi\,, \tag{19}$$

with

$$G_s(\xi) := \left(F_r(\xi)+\sqrt{-1}\ F_i(\xi)\right)e^{\sqrt{-1}\ \Lambda_s(\xi)} + \left(F_r(\xi)-\sqrt{-1}\ F_i(\xi)\right)e^{-\sqrt{-1}\ \Lambda_s(\xi)}$$

$$= 2F_r(\xi)\cos\bigl(\Lambda_s(\xi)\bigr) - 2F_i(\xi)\sin\bigl(\Lambda_s(\xi)\bigr)\,,$$

where we set $\Lambda_s(\xi) := 2\,\xi^2\bigl(\frac{B_s^2}{s^3}\,U_i(\xi)+\frac{B_s'-x}{s^2}+\frac{w^2+\beta^2}{2s}\,V_i(\xi)\bigr)$.

Note that the functions $F_r, F_i, U_r, U_i, V_r, V_i, G_s$ are all even.

6 Small time asymptotics for the density $q_s(w,\beta,x,\zeta,z)$

We shall here use the expression (19) for $q_s(w,\beta,x,\zeta,z)$, to derive its asymptotics as $s\searrow 0$, proceeding by adapting the saddle-point method (see [Co] for example). This will require the following asymptotics for the auxiliary function entering that expression.

6.1 Auxiliary asymptotics

As $\xi\to\infty$ we have $U_r(\xi) = 2\xi+4+\frac{4}{\xi}+\mathcal{O}(\frac{1}{\xi^3})$; $U_i(\xi) = 1+\mathcal{O}(\frac{1}{\xi})$; $V_r(\xi) = \frac{\xi}{2}[1+\mathcal{O}(e^{-\xi})]$; $V_i(\xi) = \frac{1+\mathcal{O}(e^{-\xi})}{4\xi}$; $F(2\xi^2) = \mathcal{O}\bigl(\frac{e^{-\xi}}{\xi}\bigr)$.

Then near 0 we successively have:

$$U_r(\xi) = \frac{2\,\xi^3\times\frac{\xi^3}{6}\bigl(1-\frac{17}{420}\xi^4+\mathcal{O}(\xi^8)\bigr)}{\bigl(\frac{\xi^5}{30}\bigr)^2+\bigl(\frac{\xi^3}{6}\bigl(1-\frac{17}{420}\xi^4\bigr)\bigr)^2+\mathcal{O}(\xi^{10})} = \frac{\frac{\xi^6}{3}\bigl(1-\frac{17}{420}\xi^4+\mathcal{O}(\xi^8)\bigr)}{\frac{\xi^6}{36}\bigl(1-\frac{17}{210}\xi^4\bigr)+\mathcal{O}(\xi^{10})} = 12+\tfrac{17}{35}\,\xi^4+\mathcal{O}(\xi^8)\,;$$

$$U_i(\xi) = \frac{\frac{\xi^6}{30}\bigl(1-\frac{113}{648}\xi^4+\mathcal{O}(\xi^8)\bigr)}{\frac{\xi^6}{36}\bigl(1-\frac{17}{210}\xi^4+\mathcal{O}(\xi^8)\bigr)} = \tfrac{6}{5}-\tfrac{2119}{18900}\,\xi^4+\mathcal{O}(\xi^8)\,;$$

$$V_i(\xi) = \tfrac{1}{12} - \tfrac{\xi^4}{756} + \mathcal{O}(\xi^8)\,;\ V_r(\xi) = 1 + \tfrac{\xi^4}{180} + \mathcal{O}(\xi^8)\,;$$

$$F_r(\xi) = \frac{6}{\xi^4} - \tfrac{620659}{135600} + \mathcal{O}(\xi^4)\,;\ F_i(\xi) = \frac{1}{5} + \mathcal{O}(\xi^4)\,;$$

$$\Lambda_s(\xi) \;=\; 2\left[\frac{B_s^2}{s^3}\big[\tfrac{6}{5} - \tfrac{2119}{18900}\,\xi^4 + \mathcal{O}(\xi^8)\big] + \frac{B_s' - x}{s^2} + \tfrac{w^2+\beta^2}{2s}\big[\tfrac{1}{12} - \tfrac{1}{756}\,\xi^4 + \mathcal{O}(\xi^8)\big]\right]\xi^2$$

$$=\; 2\mu_s\,\xi^2 - \left[\tfrac{2119}{9450}\tfrac{B_s^2}{s^3} + \tfrac{1}{63}\,\tfrac{w^2+\beta^2}{12\,s}\right]\big[\xi^6 + \mathcal{O}(\xi^{10})\big] \;=\; 2\mu_s\,\xi^2\big[1 + \mathcal{O}(\xi^4)\big],$$

where we have set

$$\mu_s \;:=\; \frac{6B_s^2}{5\,s^3} + \frac{B_s' - x}{s^2} + \frac{w^2+\beta^2}{24\,s}\,. \qquad \text{Let also} \qquad \nu_s := \frac{17\,B_s^2}{35\,s^3} + \frac{w^2+\beta^2}{360\,s}\,, \tag{20}$$

so that

$$e^{-\frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]} = e^{-\frac{B_s^2}{s^3}\left[\frac{17}{35}\,\xi^4 + \mathcal{O}(\xi^8)\right] - \frac{w^2+\beta^2}{2s}\left[\frac{1}{180}\,\xi^4 + \mathcal{O}(\xi^8)\right]} = e^{-\nu_s\,[\xi^4 + \mathcal{O}(\xi^8)]}.$$

It is not difficult to see that actually $V_r(\xi) \geq 1$ and $U_r(\xi) \geq 12$, for any real ξ.

Furthermore,

$$U_r(\xi) - 2\xi - 4 = \frac{4\xi\big(1 - \frac{2(e^{-\xi} - \sin\xi)}{\operatorname{ch}\xi + \cos\xi}\big) - 8\big(\frac{\operatorname{ch}\xi - \cos\xi}{\operatorname{ch}\xi + \cos\xi}\big) - 2\,\xi^3\big(\frac{e^{-\xi} + \cos\xi + \sin\xi}{\operatorname{ch}\xi + \cos\xi}\big) - 4\,\xi^2\big(\frac{e^{-\xi} + \cos\xi - \sin\xi}{\operatorname{ch}\xi + \cos\xi}\big)}{(\xi-1)^2 + 1 + 2\xi\big(\frac{e^{-\xi} + \cos\xi - \sin\xi}{\operatorname{ch}\xi + \cos\xi}\big) - \frac{4\,\cos\xi}{\operatorname{ch}\xi + \cos\xi}}$$

$$\Big(=\ \tfrac{4\xi - 8 + \mathcal{O}(\xi^3 e^{-\xi})}{\xi^2 - 2\xi + 2 + \mathcal{O}(\xi\,e^{-\xi})}\Big)$$

$$> \frac{4\xi - 8 - \frac{4(\xi+1)^3}{\operatorname{ch}\xi + \cos\xi}}{(\xi-1)^2 + 1 + \frac{4(\xi+1)}{\operatorname{ch}\xi + \cos\xi}} > \frac{4\xi - 8 - 9(\xi+1)^3 e^{-\xi}}{(\xi-1)^2 + 1 + 9(\xi+1)e^{-\xi}} > \frac{4\xi - 9}{\xi^2 - 2\xi + 2.2} \geq \frac{2}{\xi} \times \frac{2\pi - 9}{\pi - 1 + \frac{11}{20\pi}}$$

for $\xi \geq 2\pi$, and similarly

$$V_r(\xi) - \frac{\xi}{2} = \frac{\xi\,(\cos\xi + \sin\xi - e^{-\xi})}{2\,(\operatorname{ch}\xi - \cos\xi)} > \frac{-\xi}{\operatorname{ch}\xi - \cos\xi} > -3\xi\,e^{-\xi} > \frac{-1}{20}\,,$$

so that $V_r(\xi) > \pi - \frac{1}{20} > 3$ for $\xi \geq 2\pi$.

Therefore, in a small neighbourhood $[2\pi - \varepsilon, \infty[+ \sqrt{-1}\,[-\varepsilon, \varepsilon]$ of $[2\pi, \infty[$ we shall have

$$\Re\big[U_r(\xi) - 2\xi\big] \geq 4 \quad \text{ and } \quad \Re\big[V_r(\xi)\big] \geq 3\,. \tag{21}$$

6.2 Changes of contour and saddle-point method

Note that by Lemmas 4.2.3, 5.1, (14) and the changes of variable: $\lambda = \sqrt{-1}\, y = \pm 2\sqrt{-1}\, \xi^2$, the poles of the integrand in (19) are located at $e^{\sqrt{-1}\, k\pi/2}(1+\sqrt{-1}\,)\, y_n$ and $e^{\sqrt{-1}\, k\pi/2}(1+\sqrt{-1}\,)(n+1)\,\pi$, with $n, k \in \mathbb{N}$ and $\frac{17\pi}{12} < y_0 < y_1 < \ldots$, so that this integrand is analytic at 0 with convergence radius $\sqrt{2}\,\pi$.

In particular, we may perform the following changes of contour in (19):

$$3\pi\, s^2\, \frac{q_s(w,\beta,x,\zeta,z)}{\tilde{q}_s(w,\beta,\zeta,z)} = \int_0^\infty G_s(\xi)\, e^{-\frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$= \int_0^{(1+\sqrt{-1}\,)\mu_s^{-\eta}} \big(F_r(\xi)+\sqrt{-1}\; F_i(\xi)\big)\, e^{\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$+ \int_0^{(1-\sqrt{-1}\,)\mu_s^{-\eta}} \big(F_r(\xi)-\sqrt{-1}\; F_i(\xi)\big)\, e^{-\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$+ \int_{(1+\sqrt{-1}\,)\mu_s^{-\eta}}^{(1+\sqrt{-1}\,)\mu_s^{-\eta}+2\pi} \big(F_r(\xi)+\sqrt{-1}\; F_i(\xi)\big)\, e^{\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$+ \int_{(1-\sqrt{-1}\,)\mu_s^{-\eta}}^{(1-\sqrt{-1}\,)\mu_s^{-\eta}+2\pi} \big(F_r(\xi)-\sqrt{-1}\; F_i(\xi)\big)\, e^{-\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$+ \int_{(1+\sqrt{-1}\,)\mu_s^{-\eta}+2\pi}^{(1+\sqrt{-1}\,)\mu_s^{-\eta}+\infty} \big(F_r(\xi)+\sqrt{-1}\; F_i(\xi)\big)\, e^{\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$+ \int_{(1-\sqrt{-1}\,)\mu_s^{-\eta}+2\pi}^{(1-\sqrt{-1}\,)\mu_s^{-\eta}+\infty} \big(F_r(\xi)-\sqrt{-1}\; F_i(\xi)\big)\, e^{-\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$=: \; J_s^0 + \bar{J}_s^0 + J_s^\pi + \bar{J}_s^\pi + J_s^\infty + \bar{J}_s^\infty\,,$$

where $\eta > \frac{1}{4}$ will be specified further.

Note that the estimate (21) and the control $F(2\xi^2) = \mathcal{O}\big(\frac{e^{-\xi}}{\xi}\big)$ ensure the vanishing of the unmentioned limiting contribution (for any large enough fixed s, provided $\lim\limits_{s\to 0}\mu_s = \infty$) in the above changes of contour:

$$\lim_{R\to\infty} \int_R^{R\pm(1+\sqrt{-1}\,)\mu_s^{-\eta}} \big(F_r(\xi)+\sqrt{-1}\; F_i(\xi)\big)\, e^{\sqrt{-1}\;\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi \;= 0\,.$$

Now on the one hand, setting $\xi = (1+\sqrt{-1}\,)\mu_s^{-\eta}\, t$ by the above we have:

$$J_s^0 = \frac{8}{\mu_s^{6\eta}} \int_0^1 \big[F_i(\xi) - \sqrt{-1}\, F_r(\xi)\big]\, e^{\left[\sqrt{-1}\,\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]\right]}\, t^5 dt$$

$$= \frac{8}{5\mu_s^{6\eta}} \int_0^1 \left[1 + \tfrac{15\sqrt{-1}}{2\mu_s^{-4\eta} t^4} + \tfrac{620659\sqrt{-1}}{27120} + \mathcal{O}(\mu_s^{-4\eta})\right] e^{-4\left(\mu_s^{1-2\eta}\, t^2 - \nu_s \mu_s^{-4\eta}\right)\left(1+\mathcal{O}(\mu_s^{-4\eta})\right)}\, t^5\, dt$$

$$= \frac{8}{5\mu_s^{6\eta}} \big(1 + (\mu_s^{1-2\eta} + \nu_s)\mathcal{O}(\mu_s^{-4\eta})\big) \int_0^1 \left[1 + \tfrac{15\sqrt{-1}}{2\mu_s^{-4\eta} t^4} + \tfrac{620659\sqrt{-1}}{27120} + \mathcal{O}(\mu_s^{-4\eta})\right] e^{-4\mu_s^{1-2\eta} t^2}\, t^5\, dt\,,$$

so that for $\frac{1}{4} < \eta < \frac{1}{2}$:

$$J_s^0 + \bar{J}_s^0 \;=\; \frac{16}{5\mu_s^{6\eta}} \big(1 + \mathcal{O}(\nu_s \mu_s^{-4\eta})\big) \int_0^1 e^{-4\mu_s^{1-2\eta}\, t^2}\, t^5\, dt$$

$$=\; \frac{1 + \mathcal{O}(\nu_s\, \mu_s^{-4\eta})}{40\, \mu_s^3} \int_0^{4\mu_s^{1-2\eta}} e^{-u}\, u^2 du \;=\; \frac{1 + \mathcal{O}(\mu_s^{1-4\eta})}{20\, \mu_s^3}\,,$$

provided $\lim_{s\to 0} \mu_s = \infty$ and $\nu_s = \mathcal{O}(\mu_s)$.

Then we use the saddle-point method (as described in [Co]) to deal with the intermediate part $(J_s^\pi + \bar{J}_s^\pi)$. By the above we know that we have

$$U_i(\xi) = \tfrac{6}{5} + \xi^4\, \tilde{U}_i(\xi)\,,\ V_i(\xi) = \tfrac{1}{12} + \xi^4\, \tilde{V}_i(\xi)\,,\ U_r(\xi) - 12 = \xi^4\, \tilde{U}_r(\xi)\,,\ V_r(\xi) - 1 = \xi^4\, \tilde{V}_r(\xi)\,,$$

with even functions $\tilde{U}_i\,, \tilde{V}_i\,, \tilde{U}_r\,, \tilde{V}_r$ that are analytic outside the sequence $\big\{e^{\sqrt{-1}\ k\pi/2}(1+\sqrt{-1}\,)\, y_n \,\big|\, n,k \in \mathbb{N},\ \frac{17\pi}{12} < y_0 < y_1 < \ldots\big\}$ and then converge in the compact disc (centred at 0) of radius $\frac{17\pi}{6\sqrt{2}} > 2\pi$. (Note that the proof of Lemma 5.1 shows that the points $e^{\sqrt{-1}\,k\pi/2}(1+\sqrt{-1}\,)(n+1)\,\pi$ are poles only for the functions F_r, F_i .) Therefore we can write the phase as follows:

$$\varphi(\xi) :=\ \sqrt{-1}\ \Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi) - 12] - \frac{w^2+\beta^2}{2s}[V_r(\xi) - 1]$$

$$=\ 2\sqrt{-1}\ \mu_s\, \xi^2 - \Big(\tfrac{B_s^2}{s^3}\,[\tilde{U}_r(\xi) - 2\sqrt{-1}\ \tilde{U}_i(\xi)] + \tfrac{w^2+\beta^2}{2s}\,[\tilde{V}_r(\xi) - 2\sqrt{-1}\ \tilde{V}_i(\xi)]\Big)\, \xi^4\,.$$

Setting $M := \max\big\{|w(\xi)| \,\big|\, \xi \in \mathbb{C}, |\xi| = \frac{17\pi}{6\sqrt{2}}\big\}$, by Cauchy's inequality, for $|\xi| \le \pi + \frac{17\pi}{12\sqrt{2}}$ we have

$$\big|\varphi(\xi) - 2\sqrt{-1}\ \mu_s\, \xi^2\big| \;\le\; \frac{M\, |\xi|^4}{\frac{289\pi^2}{72}\big(\frac{289\pi^2}{72} - |\xi|^2\big)} \;\le\; \frac{4\,(72)^2\, M\, |\xi|^4}{289(579 - 408\sqrt{2})\, \pi^4} \;<\; 1156\, M.$$

As before near 0 we have $\big(F_r(\xi) + \sqrt{-1}\ F_i(\xi)\big)\xi^4 = 6 + \mathcal{O}(\xi^4)$. Hence for small s

$$J_s^\pi \;=\; \int_{(1+\sqrt{-1}\,)\mu_s^{-\eta}}^{(1+\sqrt{-1}\,)\mu_s^{-\eta}+2\pi} \big(F_r(\xi) + \sqrt{-1}\ F_i(\xi)\big)\, e^{\sqrt{-1}\ \Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$= \int_0^\pi \Big[6 + \mathcal{O}\big(|(1+\sqrt{-1}\,)\mu_s^{-\eta} + t|^4\big)\Big]\, e^{2\sqrt{-1}\ \mu_s \left((1+\sqrt{-1}\,)\mu_s^{-\eta}+t\right)^2 + \mathcal{O}(1)} \big((1+\sqrt{-1}\,)\mu_s^{-\eta} + t\big) dt$$

$$=\; \mathcal{O}(1) \int_0^\pi e^{-4\mu_s^{1-2\eta} - 4\mu_s^{1-\eta} t}\, dt \;=\; \mathcal{O}\big(e^{-4\mu_s^{1-2\eta}} / \mu_s^{1-\eta}\big)\,.$$

The same of course holds for $\bar{J}_s^\pi$.

To deal with the remaining contribution $(J_s^\infty + \bar{J}_s^\infty)$, we use the lower estimates near infinity: (21) computed above, as follows:

$$J_s^\infty = \int_{(1+\sqrt{-1})\mu_s^{-\eta}+2\pi}^{(1+\sqrt{-1})\mu_s^{-\eta}+\infty} \big(F_r(\xi) + \sqrt{-1}\, F_i(\xi)\big)\, e^{\sqrt{-1}\,\Lambda_s(\xi) - \frac{B_s^2}{s^3}[U_r(\xi)-12] - \frac{w^2+\beta^2}{2s}[V_r(\xi)-1]}\, \xi^5\, d\xi$$

$$= \int_{2\pi}^\infty \mathcal{O}\Big[\frac{e^{-t}}{t}\Big]\, e^{-\frac{B_s^2}{s^3}[2(\mu_s^{-\eta}+t)-8] - \frac{w^2+\beta^2}{s}} (1+t)^5 dt \;=\; \mathcal{O}(1)\, e^{\frac{8B_s^2}{s^3} - \frac{w^2+\beta^2}{s}} \int_{2\pi}^\infty e^{-\left[\frac{2B_s^2}{s^3}+1\right]t}\, t^4\, dt$$

$$= \mathcal{O}(1)\, e^{\frac{8B_s^2}{s^3} - \frac{w^2+\beta^2}{s}} \big(\tfrac{2B_s^2}{s^3}+1\big)^{-5} \int_{2\pi\left[\frac{2B_s^2}{s^3}+1\right]}^\infty e^{-t}\, t^4\, dt \;=\; \mathcal{O}\Big[\big(\tfrac{2B_s^2}{s^3}+1\big)^{-1}\Big]\, e^{-4(\pi-2)\frac{B_s^2}{s^3} - \frac{w^2+\beta^2}{s}},$$

and the same of course holds for $\bar{J}_s^\infty$.

So far, we have obtained:

$$3\pi\, s^2\, \frac{q_s(w,\beta,x,\zeta,z)}{\tilde{q}_s(w,\beta,\zeta,z)} \;=\; J_s^0 + \bar{J}_s^0 + J_s^\pi + \bar{J}_s^\pi + J_s^\infty + \bar{J}_s^\infty$$

$$= \frac{1+\mathcal{O}(\nu_s\,\mu_s^{-4\eta})}{20\,\mu_s^3} + \mathcal{O}\big(e^{-4\mu_s^{1-2\eta}}/\mu_s^{1-\eta}\big) + \mathcal{O}\Big[\big(\tfrac{2B_s^2}{s^3}+1\big)^{-1}\Big]\, e^{-4(\pi-2)\frac{B_s^2}{s^3} - \frac{w^2+\beta^2}{s}}$$

$$= \frac{1+\mathcal{O}(\mu_s^{1-4\eta})}{20\,\mu_s^3} \qquad \text{for } \tfrac{1}{4} < \eta < \tfrac{1}{2}\,,$$

provided both $\lim\limits_{s\to 0} \mu_s = \infty$ and $\nu_s = \mathcal{O}(\mu_s)$.

By (17) and (20), this condition holds as soon as both $\lim\limits_{s\to 0} \mu_s = \infty$ and $\frac{x}{s^2} \le \frac{z^2+\zeta^2}{s^3} + \frac{\mu_s}{\varepsilon}$ (for some $\varepsilon > 0$), and then also as soon as both $\lim\limits_{s\to 0} \nu_s = \infty$ and $2s\,x \le (z^2+\zeta^2) + \varepsilon s^2(w^2+\beta^2)$.

Finally, under this condition, as $s \searrow 0$ and for any positive ε, we have obtained:

$$q_s(w,\beta,x,\zeta,z) \;=\; \frac{1+\mathcal{O}\big(\mu_s^{\varepsilon-1}\big)}{60\,\pi\, s^2\,\mu_s^3} \times \tilde{q}_s(w,\beta,\zeta,z)\,.$$

The result of this section (and main result) is thus the following off-diagonal equivalent.

Theorem 6.3 *As $s \searrow 0$, for any positive ε, uniformly for $x \ge 0$ and $(w,\beta,\zeta,z) \in \mathbb{R}^4$ such that:*

$$\mu_s \equiv \frac{3[(z-\frac{sw}{12})^2 + (\zeta - \frac{s\beta}{12})^2]}{5\,s^3} + \frac{w^2+\beta^2}{16\,s} - \frac{x}{s^2} \longrightarrow \infty \quad \textit{and} \quad \frac{x}{s^2} \le \frac{z^2+\zeta^2}{s^3} + \frac{\mu_s}{\varepsilon}\,, \qquad (22)$$

we have

$$q_s(w,\beta,x,\zeta,z) \;=\; \frac{1+\mathcal{O}\big(\mu_s^{\varepsilon-1}\big)}{20\,\pi^3\, s^6\,\mu_s^3} \times \exp\left[-\frac{3\,[(sw-2z)^2 + (s\beta-2\zeta)^2]}{2\,s^3} - \frac{w^2+\beta^2}{2\,s}\right].$$

An alternative condition (to (22) above) guaranteeing this asymptotic equivalent is

$$\frac{(sw-2z)^2+(s\beta-2\zeta)^2}{s^3}+\frac{w^2+\beta^2}{s}\longrightarrow\infty\quad and\quad 2s\,x\le(z^2+\zeta^2)+\varepsilon s^2(w^2+\beta^2)\,. \tag{23}$$

Remark 6.4 Theorem 6.3 rather precisely yields the small time asymptotic behaviour of the heat kernel of the second chaos approximation (Y_s) to the Dudley relativistic diffusion (X_s). But this is not enough to derive any small time asymptotics for the density $p_s(\lambda,\mu,x,y,z)$ of the original process $X_s\equiv(\lambda_s\,,\,\mu_s\,,\,x_s\,,\,y_s\,,\,z_s)$, even for fixed (λ,μ,x,y,z). The reason is that the computations performed in Section 4 above cannot work beyond the second chaos, so that Section 3 cannot yield a sufficiently precise control on the gap between both tangent processes (X_s) and (Y_s). To be more specific, Section 3 and Theorem 6.3 heuristically yield :

$$p_s(w,\beta,x,\zeta,z)\approx\mathbb{P}\big[\lambda_s=w\,,\,\mu_s=\beta\,,\,x_s=x\,,\,y_s=\zeta\,,\,z_s=z\big]\approx$$

$$\mathbb{P}\Big[(w_s\,,\,\beta_s\,,\,A_s\,,\,\zeta_s\,,\,\bar z_s)=(w,\beta,x,\zeta,z)+\mathcal{O}(R)(s^{3/2},s^{3/2},s^{5/2},s^{5/2},s^{5/2})\Big]+\mathcal{O}\big(e^{-R^\kappa/c}\big)$$

$$\approx\frac{\exp\left[-\frac{3\left[\left(sw-2z+\mathcal{O}(s^{5/2})\right)^2+\left(s\beta-2\zeta+\mathcal{O}(s^{5/2})\right)^2\right]}{2\,s^3}-\frac{w^2+\beta^2+\mathcal{O}(s^{3/2})}{2\,s}\right]}{20\,\pi^3\,s^6\,\mu_s\big(w+\mathcal{O}(s^{3/2}),\beta+\mathcal{O}(s^{3/2}),x+\mathcal{O}(s^{5/2}),\zeta+\mathcal{O}(s^{5/2}),z+\mathcal{O}(s^{5/2})\big)^3}+\mathcal{O}\big(e^{-R^\kappa/c}\big)$$

$$=\exp\left[-\frac{3\left[(sw-2z)^2+(s\beta-2\zeta)^2\right]}{2\,s^3}-\frac{w^2+\beta^2}{2\,s}+\mathcal{O}(s^{-1/2})\right]+\mathcal{O}\big(e^{-R^\kappa/c}\big)\,,$$

which were not too bad only if the additive term $\mathcal{O}\big(e^{-R^\kappa/c}\big)$ were not there to ruin such estimate. Indeed, even taking $R\asymp s^{-\gamma}$ would only control this correction term by (at best) $e^{-s^{-2\gamma/3}}$, which would be significant only for $\gamma\ge 9/2$, so that the remaining information would then reduce to nothing.

7 Proofs of some technical results

We gather here the rather technical proofs of Lemmas 3.1 and 4.1.1 and Proposition 5.2.

Proof of Lemma 3.1 (i) Equation (4) entails that for small proper time s we have:

$$\lambda_s=w_s+\tfrac12\int_0^s\mathrm{th}\big(w_\tau+o(\tau)\big)d\tau=w_s+\tfrac12\int_0^s w_\tau\,d\tau+o(s^2)=\,w_s+o(s^{3/2-\varepsilon})$$

$$=w_s+\tfrac12\int_0^s\mathrm{th}\big(w_\tau+o(\tau^{3/2-\varepsilon})\big)d\tau=w_s+\tfrac12\int_0^s\big(w_\tau+o(\tau^{3/2-\varepsilon})\big)d\tau=w_s+\tfrac12\int_0^s w_\tau\,d\tau+o(s^{5/2-\varepsilon}).$$

Then

$$\mathrm{ch}\,\lambda_s=\mathrm{ch}\big[w_s+o(s^{3/2-\varepsilon})\big]=1+\tfrac12\,w_s^2+o(s^{2-\varepsilon})=1+o(s^{1-\varepsilon})\,,$$

and by Equation (5) we have:

$$\dot z_s = \operatorname{sh}\lambda_s = \operatorname{sh}\big[w_s + o(s^{3/2-\varepsilon})\big] = w_s + o(s^{3/2-\varepsilon})\,.$$

$$\Big(\text{Also,}\quad \dot z_s = \textstyle\int_0^s \sqrt{1+\dot z_\tau^2}\,dw_\tau + \int_0^s \dot z_\tau\,d\tau\,.\Big)$$

Similarly,

$$\mu_s = \int_0^s \frac{d\beta_\tau}{\operatorname{ch}\,[w_\tau + o(\tau^{3/2-\varepsilon})]} = \int_0^s \big(1 - o(\tau^{1-\varepsilon})\big)d\beta_\tau = \beta_s + o(s^{3/2-\varepsilon})\,,$$

$$\operatorname{ch}\mu_s = 1 + o(s^{1-\varepsilon})\,,\quad \operatorname{sh}\mu_s = \beta_s + o(s^{3/2-\varepsilon})\,.$$

$$\Big(\text{Also,}\quad \operatorname{sh}\mu_s = \textstyle\int_0^s \sqrt{\frac{1+\operatorname{sh}^2\mu_\tau}{1+\dot z_\tau^2}}\,d\beta_\tau + \int_0^s \frac{\operatorname{sh}\mu_\tau}{2(1+\dot z_\tau^2)}\,d\tau\,.\Big)$$

Finally, the result follows at once from :

$$x_s = \int_0^s \big[1 + \tfrac12\,w_\tau^2 + o(\tau^{2-\varepsilon})\big]\big[1 + \tfrac12\,\beta_\tau^2 + o(\tau^{2-\varepsilon})\big]d\tau = s + \tfrac12 \int_0^s \big[\beta_\tau^2 + w_\tau^2\big]d\tau + o(s^{3-\varepsilon})\,;$$

$$y_s = \int_0^s \big[1 + o(\tau^{1-\varepsilon})\big]\big[\beta_\tau + o(\tau^{3/2-\varepsilon})\big]d\tau = \int_0^s \beta_\tau\,d\tau + o(s^{5/2-\varepsilon})\,;\ z_s = \int_0^s w_\tau\,d\tau + o(s^{5/2-\varepsilon})\,.$$

$$\Big(\text{More precisely,}\quad z_s = \textstyle\int_0^s w_\tau\,d\tau + \int_0^s\!\int_0^\tau w_u\,du\,d\tau + \frac12\int_0^s\!\int_0^\tau w_u^2\,dw_u\,d\tau + o(s^{7/2-\varepsilon})\,.\Big)$$

(ii) For any $s\in[0,1]$ we almost surely have : $|\lambda_s| \le \sup_{[0,s]}|w| + \frac12\int_0^s |\lambda_\tau|\,d\tau$, whence by Gronwall's Lemma : $\sup_{[0,s]}|\lambda| \le \sup_{[0,s]}|w|\times e^{s/2}$. Then for $0\le s\le 2\log 2$:

$$\Lambda_s := \sup_{[0,s]}|\lambda - w| \le \tfrac12 \int_0^s \sup_{[0,\tau]}|w|\times e^{\tau/2}d\tau \le \int_0^s \sup_{[0,\tau]}|w|\,d\tau\,.$$

Hence for $R\ge 1$ and $0\le s\le 2\log 2$:

$$\mathbb{P}\Big[\sup_{0\le t\le s}|\Lambda_t| \ge R\,s^{\frac32}\Big] \le \mathbb{P}\Big[\int_0^1 \sup_{[0,\tau]}|w|\,d\tau \ge R\Big] \le \mathbb{P}\Big[\sup_{[0,1]}|w| \ge R\Big] \le 4\,\mathbb{P}[w_1\ge R] \le 2\,e^{-\frac{R^2}{2}}.$$

Then $\beta_s - \mu_s = 2\displaystyle\int_0^s \frac{\operatorname{sh}^2(\lambda_\tau/2)}{\operatorname{ch}\lambda_\tau}\,d\beta_\tau \equiv 2\,B\Big[\int_0^s \frac{\operatorname{sh}^4(\lambda_\tau/2)}{\operatorname{ch}^2\lambda_\tau}\,d\tau\Big]$, so that

$$\mathbb{P}\Big[\sup_{0\le t\le s}|\beta_t - \mu_t| \ge R\,s^{\frac32}\Big] \le 2\,\mathbb{E}\Big[\exp\Big(-\frac{R^2 s^3}{8}\Big/\int_0^s \frac{\operatorname{sh}^4(\lambda_\tau/2)}{\operatorname{ch}^2\lambda_\tau}\,d\tau\Big)\Big]$$

$$\le 2\,\mathbb{E}\Big[\exp\Big(-2R^2 s^3\Big/\int_0^s \lambda_\tau^4\,d\tau\Big)\Big] \le 2\,\mathbb{E}\Big[\exp\Big(-\frac{R^2}{8\sup_{[0,1]}|w|^4}\Big)\Big]$$

$$= 2\int_0^1 \mathbb{P}\Big[-\frac{R^2}{8\sup\limits_{[0,1]}|w|^4} > \log y\Big] dy = 2\int_0^1 \mathbb{P}\Big[\sup_{[0,1]}|w|^4 > \frac{-R^2}{8\log y}\Big] dy$$

$$\leq 4\int_0^1 e^{-\frac{1}{2}\sqrt{\frac{-R^2}{8\log y}}}\,dy = 4\int_0^1 e^{\frac{-R}{4\sqrt{2\log(1/y)}}}\,dy = 4\int_0^\infty e^{\frac{-R}{4\sqrt{2t}}-t}\,dt \;\leq\; 132\, e^{-R^{2/3}/(32)^{1/3}}.$$

Then $$\Big| z_s - \int_0^s w_\tau\, d\tau\Big| \;\leq \int_0^s \Big|\mathrm{sh}\big(w_\tau + (\lambda_\tau - w_\tau)\big) - w_\tau\Big|\, d\tau$$

$$= \int_0^s \Big|\mathrm{sh}\, w_\tau - w_\tau + 2\,\mathrm{sh}\big[\tfrac{\lambda_\tau - w_\tau}{2}\big]\,\mathrm{ch}\big[\tfrac{\lambda_\tau + w_\tau}{2}\big]\Big|\, d\tau \;\leq \int_0^s \Big(R^3\tau^{3/2} + 2\,\mathrm{sh}\big[\tfrac{R\,\tau^{3/2}}{2}\big]\,\mathrm{ch}\big[2R\sqrt{\tau}\,\big]\Big) d\tau$$

$$\leq \int_0^s (R^3 + 2R)\,\tau^{3/2} d\tau \;\leq\; R^3 s^{5/2} \quad \text{for } 0\leq s\leq s_R > 0\,, \quad \text{with probability } 1 - \mathcal{O}(e^{-R^2/2}).$$

Similarly $$\Big| y_s - \int_0^s \beta_\tau\, d\tau\Big| \;\leq \int_0^s \Big|\mathrm{ch}\big(w_\tau + (\lambda_\tau - w_\tau)\big)\,\mathrm{sh}\big(\beta_\tau + (\mu_\tau - \beta_\tau)\big) - \beta_\tau\Big|\, d\tau$$

$$= \int_0^s \Big| \big(1 + 2\,\mathrm{sh}^2\big[\tfrac{w_\tau}{2}\big] + 2\,\mathrm{sh}\big[\tfrac{\lambda_\tau - w_\tau}{2}\big]\,\mathrm{sh}\big[\tfrac{\lambda_\tau + w_\tau}{2}\big]\big)\big(\mathrm{sh}\,\beta_\tau + 2\,\mathrm{sh}\big[\tfrac{\mu_\tau - \beta_\tau}{2}\big]\,\mathrm{ch}\big[\tfrac{\mu_\tau + \beta_\tau}{2}\big]\big) - \beta_\tau\Big|\, d\tau$$

$$\leq \int_0^s \Big(R^3\tau^{3/2} + R\,\tau^{3/2} + \mathcal{O}\big(R\,\tau^{5/2}\big)\Big) d\tau \;\leq\; R^3 s^{5/2} \quad \text{for } 0\leq s\leq s'_R > 0\,,$$

with probability $1 - \mathcal{O}(e^{-R^{2/3}/4})$. Finally, in the same way we obtain:

$$\Big| x_s - \Big(s + \tfrac{1}{2}\int_0^s \big[\beta_\tau^2 + w_\tau^2\big] d\tau\Big)\Big| \leq \int_0^s \Big|\mathrm{ch}\big(w_\tau + (\lambda_\tau - w_\tau)\big)\,\mathrm{ch}\big(\beta_\tau + (\mu_\tau - \beta_\tau)\big) - 1 - \tfrac{1}{2}\big[\beta_\tau^2 + w_\tau^2\big]\Big|\, d\tau$$

$$= \int_0^s \Big|\big(\mathrm{ch}\, w_\tau + 2\,\mathrm{sh}\big[\tfrac{\lambda_\tau - w_\tau}{2}\big]\,\mathrm{sh}\big[\tfrac{\lambda_\tau + w_\tau}{2}\big]\big)\big(\mathrm{ch}\,\beta_\tau + 2\,\mathrm{sh}\big[\tfrac{\mu_\tau - \beta_\tau}{2}\big]\,\mathrm{sh}\big[\tfrac{\mu_\tau + \beta_\tau}{2}\big]\big) - 1 - \tfrac{1}{2}\big[\beta_\tau^2 + w_\tau^2\big]\Big|\, d\tau$$

$$\leq \int_0^s \Big(\big|2\,\mathrm{sh}^2\big[\tfrac{w_\tau}{2}\big] - \tfrac{w_\tau^2}{2}\big| + \big|2\,\mathrm{sh}^2\big[\tfrac{\beta_\tau}{2}\big] - \tfrac{\beta_\tau^2}{2}\big| + \mathcal{O}\big(R^2\tau^2\big)\Big) d\tau$$

$$\leq \int_0^s \Big(R^4\tau^2 + \mathcal{O}\big(R^2\tau^2\big)\Big) d\tau \;\leq\; R^4 s^3 \;\leq\; s^{5/2} \quad \text{for } 0\leq s\leq s''_R > 0\,,$$

with probability $1 - \mathcal{O}(e^{-R^{2/3}/4})$. In particular we can take $\kappa = 2/9$ in the statement. $\diamond$

Proof of Lemma 4.1.1 Let us use $\big($[Y], Chapter (2) "The laws of some quadratic functionals of Brownian motion"$\big)$, considering for any $b > 0$ the exponential martingale

$$M_s^b := \exp\Big(-\tfrac{b}{2}\,(w_s^2 - w_0^2 - s) - \tfrac{b^2}{2}\int_0^s w_\tau^2\, d\tau\Big) = \exp\Big(-b\int_0^s w_\tau\, dw_\tau - \tfrac{b^2}{2}\int_0^s w_\tau^2\, d\tau\Big),$$

and the new probability $\mathbb{P}^b$ having on $\mathcal{F}_s$ density M_s^b with respect to $\mathbb{P}$. As noticed in $\big($[Y], (2.1.1)$\big)$, Girsanov's Theorem yields a real $(\mathbb{P}^b, \mathcal{F}_s)$-Brownian motion $(B_u\,,\, 0\leq u\leq s)$ such

that $w_u = w_0 + B_u - b\int_0^u w_\tau\,d\tau$, which means that under $\mathbb{P}^b$, w has become an Ornstein-Uhlenbeck process, alternatively expressed by $w_u = e^{-b\,u}\Big(w_0 + \int_0^u e^{b\,\tau}\,dB_\tau\Big)$. Therefore

$$\mathbb{E}_0\Big[\exp\Big(\sqrt{-1}\int_0^s (a+c\,\tau)\,dw_\tau - \tfrac{b^2}{2}\int_0^s w_\tau^2\,d\tau\Big)\Big]$$

$$= e^{-b\,s/2}\times\mathbb{E}_0^b\Big[\exp\Big(\sqrt{-1}\int_0^s (a+c\,\tau)\,dw_\tau + b\,w_s^2/2\Big)\Big]$$

on the one hand, and on the other hand for any test-function f on $\mathbb{R}$:

$$\int_0^s f(u)\,dw_u = \int_0^s f(u)\Big[dB_u - b\,e^{-b\,u}\Big(\int_0^u e^{b\,\tau}\,dB_\tau\Big)du\Big]$$

$$= \int_0^s \Big(f(\tau) - b\,e^{b\,\tau}\int_\tau^s f(u)\,e^{-b\,u}du\Big)dB_\tau\,,$$

so that

$$\mathbb{E}_0^b\Big[\Big(\int_0^s f(\tau)\,dw_\tau\Big)^2\Big] = \int_0^s \Big[f(\tau) - b\,e^{b\tau}\int_\tau^s f(u)\,e^{-b\,u}du\Big]^2 d\tau\,.$$

Taking $f(\tau) = a + c\,\tau$, we obtain

$$\mathbb{E}_0^b\Big[\Big(a\,w_s + c\int_0^s \tau\,dw_\tau\Big)^2\Big] = \int_0^s \Big[a + c\,\tau + (a+c/b)(e^{b(\tau-s)}-1) + c\,(s\,e^{b(\tau-s)} - \tau)\Big]^2 d\tau$$

$$= b^{-2}\int_0^s \Big[(ab + c + bc\,s)\,e^{b(\tau-s)} - c\Big]^2 d\tau$$

$$= b^{-2}\int_0^s \Big[(ab+c+bc\,s)^2\,e^{2b(\tau-s)} - 2c\,(ab+c+bc\,s)\,e^{b(\tau-s)} + c^2\Big]d\tau$$

$$= b^{-3}\Big[\tfrac{1}{2}\big(b\,a + (bs+1)c\big)^2(1-e^{-2bs}) - 2c\,\big(b\,a + (bs+1)c\big)(1-e^{-bs}) + bs\,c^2\Big]$$

$$= \tfrac{1-e^{-2bs}}{2\,b}\,a^2 + \tfrac{bs-1+2e^{-b\,s}-(bs+1)e^{-2bs}}{b^2}\,ac + \tfrac{b^2s^2-3+4\,(bs+1)\,e^{-bs}-(bs+1)^2\,e^{-2bs}}{2\,b^3}\,c^2\,.$$

This yields the covariance matrix of the $\mathbb{P}_0^b$-Gaussian variable $\Big(w_s\,,\int_0^s \tau\,dw_\tau\Big)$, hence its joint law. Namely the covariance matrix under $\mathbb{P}_0^b$ of $\Big(\sqrt{2b}\,w_s\,,\sqrt{2b^3}\int_0^s \tau\,dw_\tau\Big)$ is

$$K_{bs} = \begin{pmatrix} 1-e^{-2bs} & bs-1+2e^{-bs}-(bs+1)e^{-2bs} \\ bs-1+2e^{-bs}-(bs+1)e^{-2bs} & b^2s^2-3+4\,(bs+1)\,e^{-bs}-(bs+1)^2\,e^{-2bs} \end{pmatrix},$$

and its determinant is $\delta_{bs} := 2(bs-2) + 8\,e^{-b\,s} - 2(bs+2)\,e^{-2b\,s}$, which increases with $bs>0$ and then does not vanish. Therefore the density of $\left(\sqrt{2b}\,w_s\,,\ \sqrt{2b^3}\int_0^s \tau\,dw_\tau\right)$ is

$$(u,v) \longmapsto \frac{1}{2\pi\sqrt{\delta_{bs}}}\,\exp\left[\frac{-1}{2\,\delta_{bs}}\Big(\alpha_{bs}\,u^2 + 2\,\gamma_{bs}\,uv + \left(1-e^{-2bs}\right)v^2\Big)\right],$$

with $\alpha_x := x^2 - 3 + 4\,(x+1)\,e^{-x} - (x+1)^2\,e^{-2x}$ and $\gamma_x := 1 - x - 2e^{-x} + (x+1)e^{-2x}$, so that $\delta_x = (1-e^{-2x})\,\alpha_x - \gamma_x^2$. Hence

$$\mathbb{E}_0^b\left[\exp\left(\sqrt{-1}\int_0^s (a+c\,\tau)\,dw_\tau + b\,w_s^2/2\right)\right]$$

$$= \frac{1}{2\pi\sqrt{\delta_{bs}}}\int_{\mathbb{R}^2} e^{\frac{a\sqrt{-1}}{\sqrt{2b}}\,u + \frac{c\sqrt{-1}}{\sqrt{2b^3}}\,v + u^2/4}\exp\left[\frac{-1}{2\,\delta_{bs}}\Big(\alpha_{bs}\,u^2 + 2\,\gamma_{bs}\,uv + \left(1-e^{-2bs}\right)v^2\Big)\right]du\,dv$$

$$= \frac{1}{\sqrt{2\pi(1-e^{-2bs})}}\int_{\mathbb{R}} e^{\frac{a\sqrt{-1}}{\sqrt{2b}}\,u + \frac{\delta_{bs}}{2(1-e^{-2bs})}\left(\frac{c\sqrt{-1}}{\sqrt{2b^3}} - \frac{\gamma_{bs}}{\delta_{bs}}\,u\right)^2 + \left(\frac14 - \frac{\alpha_{bs}}{2\,\delta_{bs}}\right)u^2}\,du$$

$$= \frac{e^{\frac{-\delta_{bs}\,c^2}{4(1-e^{-2bs})b^3}}}{\sqrt{2\pi(1-e^{-2bs})}}\int_{\mathbb{R}} e^{\left(\frac{a\sqrt{-1}}{\sqrt{2b}} - \frac{c\sqrt{-1}\,\gamma_{bs}}{(1-e^{-2bs})\sqrt{2b^3}}\right)u - \left(\frac{\alpha_{bs}}{\delta_{bs}} - \frac12 - \frac{\gamma_{bs}^2}{(1-e^{-2bs})\delta_{bs}}\right)u^2/2}\,du$$

$$= \frac{e^{\frac{-\delta_{bs}\,c^2}{4(1-e^{-2bs})b^3}}}{\sqrt{2\pi(1-e^{-2bs})}}\int_{\mathbb{R}} e^{\left(a - \frac{c\,\gamma_{bs}}{b\,(1-e^{-2bs})}\right)\frac{\sqrt{-1}}{\sqrt{2b}}\,u - \left(\frac{1+e^{-2bs}}{1-e^{-2bs}}\right)u^2/4}\,du$$

$$= \frac{2\,e^{\frac{-\delta_{bs}\,c^2}{4(1-e^{-2bs})b^3}}}{\sqrt{2(1+e^{-2bs})}} \times e^{\frac{-1}{2b^3}\times\frac{1-e^{-2bs}}{1+e^{-2bs}}\left(ab - \frac{c\,\gamma_{bs}}{1-e^{-2bs}}\right)^2}$$

$$= \frac{e^{bs/2}}{\sqrt{\mathrm{ch}(bs)}}\,\exp\left[\frac{-1}{4b^3}\left(\frac{\delta_{bs}\,c^2}{1-e^{-2bs}} + 2\,\frac{1-e^{-2bs}}{1+e^{-2bs}}\left(a\,b - \frac{\gamma_{bs}\,c}{1-e^{-2bs}}\right)^2\right)\right]$$

$$= \frac{e^{bs/2}}{\sqrt{\mathrm{ch}(bs)}}\,\exp\left[\frac{-1}{4b^3(1+e^{-2bs})}\Big(2\,(1-e^{-2bs})\,b^2a^2 + (2\alpha_{bs} - \delta_{bs})\,c^2 - 4\,b\,\gamma_{bs}\,ac\Big)\right].$$

Therefore for any $b,s>0$ and real a,c we obtain:

$$\mathbb{E}_0\left[\exp\left(\sqrt{-1}\int_0^s (a+c\,\tau)\,dw_\tau - \tfrac{b^2}{2}\int_0^s w_\tau^2\,d\tau\right)\right]$$

$$= \frac{1}{\sqrt{\mathrm{ch}(bs)}}\,\exp\left[\frac{-1}{4\,b^3(1+e^{-2bs})}\Big(2\,(1-e^{-2bs})\,b^2a^2 + (2\alpha_{bs} - \delta_{bs})\,c^2 - 4\,\gamma_{bs}\,b\,ac\Big)\right].$$

Then taking $a = r + c\,s$, we get:

$$\mathbb{E}_0\left[\exp\left(\sqrt{-1}\,\left[r\,w_s + c\int_0^s w_\tau\,d\tau\right] - \tfrac{b^2}{2}\int_0^s w_\tau^2\,d\tau\right)\right]$$

$$= \mathbb{E}_0\Big[\exp\Big(\sqrt{-1}\int_0^s (r+c\,s-c\,\tau)\,dw_\tau - \tfrac{b^2}{2}\int_0^s w_\tau^2\;d\tau\Big)\Big]$$

$$= \frac{1}{\sqrt{\mathrm{ch}(bs)}}\,\exp\Big[-\,\frac{2(1-e^{-2bs})\,b^2(r+c\,s)^2 + (2\alpha_{bs}-\delta_{bs})\,c^2 + 4\,\gamma_{bs}\,b\,(r+c\,s)\,c}{4\,b^3(1+e^{-2bs})}\Big]$$

$$= \frac{1}{\sqrt{\mathrm{ch}(bs)}}\exp\Big[-\,\frac{(1-e^{-2bs})\,b^2(r^2+2s\,rc) + \big(bs\,(1+e^{-2bs})-(1-e^{-2bs})\big)c^2 + 2b\,\gamma_{bs}\,rc}{2\,b^3(1+e^{-2bs})}\Big]$$

$$= \frac{1}{\sqrt{\mathrm{ch}(bs)}}\,\exp\Big[-\,\frac{\mathrm{th}(bs)}{2\,b}\,r^2 - \frac{bs-\mathrm{th}(bs)}{2\,b^3}\,c^2 - 2\,\frac{\mathrm{sh}^2(bs/2)}{b^2\,\mathrm{ch}(bs)}\,rc\Big].\ \diamond$$

Proof of Proposition 5.2 Denote by $\lambda_0 \in \mathbb{R}_+$ the abscissa of convergence of the integral, so that the map $\lambda \mapsto \int_0^\infty e^{\lambda x}\,q_1(w,\beta,x,\zeta,z)\,dx$ is analytic on $\big\{\Re(\lambda)<\lambda_0\big\}$. By Proposition 4.2.2 and Lemma 5.1 it is equal to $\Phi(\lambda)$ for $\Re(\lambda)<\min\{4\pi^2,\lambda_0\}$. Hence, for any real $\lambda<\min\{4\pi^2,\lambda_0\}$ and $t\in\mathbb{R}$ we have

$$\Phi(\lambda+\sqrt{-1}\;t) = \int_0^\infty e^{\sqrt{-1}\;t\,x}\,e^{\lambda\,x}\,q_1(w,\beta,x,\zeta,z)\,dx\,.$$

Let us show now that $t\mapsto\Phi(\lambda+\sqrt{-1}\;t)$ belongs to $L^1\cap L^2(\mathbb{R})$, in order to inverse the above Fourier transform. Of course we have to deal here with the large values of $|t|$, i.e., of $|\lambda+\sqrt{-1}\;t|$ in the expression (14):

$$\Phi_{w,\beta,\zeta,z}(\lambda) = \frac{\lambda^2\,e^{B'\lambda-\frac{\sqrt{\lambda}}{4}\mathrm{cotg}(\frac{\sqrt{\lambda}}{2})(w^2+\beta^2)}}{8\pi^2\big[1-\cos(\sqrt{\lambda}\,)-(\sqrt{\lambda}/2)\sin(\sqrt{\lambda}\,)\big]}\,\exp\left[\frac{B^2\lambda}{1-\frac{2}{\sqrt{\lambda}}\,\mathrm{tg}(\frac{\sqrt{\lambda}}{2})}\right],$$

in which λ is to be replaced by $\lambda+\sqrt{-1}\,t$, and we have set $B^2 := \frac{(w-2z)^2+(\beta-2\zeta)^2}{8}$ and $B' := \frac{4wz+4\beta\zeta-w^2-\beta^2}{8} = \frac{z^2+\zeta^2}{2}-B^2$. Then for any $t\in\mathbb{R}$ we have:

$$\frac{\sqrt{\lambda+\sqrt{-1}\;t}}{2} =: \alpha+\sqrt{-1}\;b = \sqrt{\tfrac{|\lambda+\sqrt{-1}\;t|+\lambda}{8}} + \sqrt{-1}\;\mathrm{sign}(t)\sqrt{\tfrac{|\lambda+\sqrt{-1}\;t|-\lambda}{8}}$$

and

$$\mathrm{tg}\Big[\tfrac{\sqrt{\lambda+\sqrt{-1}\;t}}{2}\Big] = \frac{\mathrm{tg}\,\alpha+\sqrt{-1}\;\mathrm{th}\,b}{1-\sqrt{-1}\,\mathrm{tg}\,\alpha\,\mathrm{th}\,b} = \frac{\sin(2\alpha)+\sqrt{-1}\,\mathrm{sh}(2b)}{\mathrm{ch}(2b)+\cos(2\alpha)} = \sqrt{-1}\,\mathrm{sign}(t)+\mathcal{O}\Big(e^{-\sqrt{|t|/2}}\Big).$$

Therefore, for large $|t|$ we have:

$$\exp\left[B'(\lambda+\sqrt{-1}\;t)-(w^2+\beta^2)\,\tfrac{\sqrt{\lambda+\sqrt{-1}\;t}}{4}\,\mathrm{cotg}\big(\tfrac{\sqrt{\lambda+\sqrt{-1}\;t}}{2}\big)+\frac{B\,(\lambda+\sqrt{-1}\;t)}{1-\frac{2}{\sqrt{\lambda+\sqrt{-1}\;t}}\,\mathrm{tg}\Big(\frac{\sqrt{\lambda+\sqrt{-1}\;t}}{2}\Big)}\right]$$

$$= e^{\left(\frac{z^2+\zeta^2}{2}\right)\lambda}\exp\Big[\big(\tfrac{z^2+\zeta^2}{2}\big)\sqrt{-1}\,t+\sqrt{-1}\;\mathrm{sign}(t)\big(\tfrac{w^2+\beta^2}{4}+2B\big)\sqrt{\lambda+\sqrt{-1}\,t}\Big[1+\mathcal{O}\big(|t|^{-1/2}\big)\Big]\Big],$$

the modulus of which is

$$e^{(z^2+\zeta^2)\,\lambda/2}\ \exp\Big[-(\tfrac{w^2+\beta^2}{4}+2B^2)\sqrt{\tfrac{|\lambda+\sqrt{-1}\,t|-\lambda}{2}}+\mathcal{O}(1)\Big].$$

Moreover

$$\cos\big(\sqrt{\lambda+\sqrt{-1}\,t}\,\big)=\mathrm{ch}(2b)\cos(2\alpha)-\sqrt{-1}\ \mathrm{sh}(2b)\sin(2\alpha)$$

and

$$\sin\big(\sqrt{\lambda+\sqrt{-1}\,t}\,\big)=\mathrm{ch}(2b)\sin(2\alpha)+\sqrt{-1}\ \mathrm{sh}(2b)\cos(2\alpha)$$

entail

$$\Big|1-\cos\big(\sqrt{\lambda+\sqrt{-1}\,t}\,\big)-\big(\sqrt{\lambda+\sqrt{-1}\,t}/2\big)\sin\big(\sqrt{\lambda+\sqrt{-1}\,t}\,\big)\Big|^2$$

$$=[\mathrm{ch}(2b)-\cos(2\alpha)]^2+(\alpha^2+b^2)[\mathrm{ch}^2(2b)-\cos^2(2\alpha)]-2[\mathrm{ch}(2b)-\cos(2\alpha)][b\,\mathrm{sh}(2b)+\alpha\sin(2\alpha)]$$

$$=[\mathrm{ch}(2b)-\cos(2\alpha)-b\,\mathrm{sh}(2b)-\alpha\sin(2\alpha)]^2+[\alpha\,\mathrm{sh}(2b)-b\sin(2\alpha)]^2$$

$$=(\alpha^2+b^2)\,\mathrm{sh}^2(2b)+\mathcal{O}\big(\alpha\,\mathrm{sh}^2(2b)\big)=\Big[\tfrac{|\lambda+\sqrt{-1}\,t|}{4}+\mathcal{O}\big(\sqrt{|t|}\,\big)\Big]\,\mathrm{sh}^2\sqrt{\tfrac{|\lambda+\sqrt{-1}\,t|-\lambda}{2}}\quad\text{for large }|t|\,.$$

So far, for $\lambda<4\pi^2$ and for large $|t|$ we have: $\big|\Phi_{w,\beta,\zeta,z}(\lambda+\sqrt{-1}\,t)\big|$

$$=\ (\lambda^2+t^2)^{3/4}\,e^{(z^2+\zeta^2)\,\lambda/2}\ \exp\Big[-(\tfrac{w^2+\beta^2}{4}+2B^2+1)\sqrt{\tfrac{\sqrt{\lambda^2+t^2}-\lambda}{2}}+\mathcal{O}(1)\Big]\Big[1+\mathcal{O}\big(|t|^{-1/2}\big)\Big]$$

$$=\ \mathcal{O}\Big(e^{2\pi^2(z^2+\zeta^2)}\Big)|t|^{3/2}\exp\Big[-\big(\tfrac{w^2+\beta^2}{4}+2B^2+1+\mathcal{O}(|t|^{-1})\big)\sqrt{|t|/2}\,\Big]$$

$$=\ \mathcal{O}\Big(e^{2\pi^2(z^2+\zeta^2)}\Big)\exp\Big[-\big(\tfrac{w^2+\beta^2}{4}+2B^2+1/2\big)\sqrt{|t|/2}\,\Big].$$

Hence we can inverse the above Fourier transform for $\lambda<\min\{4\pi^2,\lambda_0\}$, and thus we obtain the wanted (15), which holds a posteriori for $\lambda<4\pi^2$:

$$e^{\lambda\,x}\,q_1(w,\beta,x,\zeta,z)=\int_{-\infty}^{\infty}e^{-\sqrt{-1}\,t\,x}\,\Phi(\lambda+\sqrt{-1}\,t)\,\frac{dt}{2\pi}\,,\ \text{ for }x>0\text{ and for }\lambda<4\pi^2.$$

Thence, for any real $\lambda<4\pi^2$, taking $\varepsilon<4\pi^2-\lambda$ for positive x we have

$$e^{\lambda\,x}\,q_1(w,\beta,x,\zeta,z)=\mathcal{O}\big(e^{-\varepsilon\,x}\big)\times\int_{-\infty}^{\infty}\big|\Phi(\lambda+\varepsilon+\sqrt{-1}\,t)\big|\,dt\ =\mathcal{O}\big(e^{-\varepsilon\,x}\big),$$

which entails the integrability of $x\mapsto e^{\lambda\,x}\,q_1(w,\beta,x,\zeta,z)$, so that finally $\lambda_0\geq 4\pi^2$. $\diamond$

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